AN INTRODUCTION ON STABILITY OF PROJECTIVE VARIETIES

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1. Stability and examples

The geometric invariant theory is a useful method to construct a moduli space or a compactified moduli space of algebraic varieties if one can describe criteria for stability and semi-stability.

Definition. Let \( V \) be a finite dimensional \( \mathbb{C} \) vector space and suppose that the special linear group \( G = \text{SL}(n, \mathbb{C}) \) operates on \( V \) (equivalently, \( V \) is a representation of \( G \)). Let \( x \in V \).

1. \( x \) is unstable if there exists a one parameter subgroup \( \lambda \) of \( G \) such that the weights of \( x \) with respect to \( \lambda \) are all positive.
2. \( x \) is semi-stable if every one parameter subgroup \( \lambda \) of \( G \) such that the weights of \( x \) with respect to \( \lambda \) are not all positive.
3. \( x \) is stable if for all non-trivial one parameter subgroup \( \lambda \) of \( G \), \( x \) has both positive and negative weights with respect to \( \lambda \).

Remark. Recall that a one parameter subgroup \( \lambda \) of \( G \) is just a homomorphism \( \lambda : \mathbb{G}_m \to G \). Such \( \lambda \) can always be diagonalized in a suitable basis:

\[
\lambda(t) = \begin{bmatrix}
t^{r_0} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & t^{r_n}
\end{bmatrix}
\]

If in this basis \( x = (x_0, \ldots, x_n) \), the set of weights of \( x \) with respect to \( \lambda \) is the set of \( r_i \) for which \( x_i \neq 0 \).

Example (0-CYCLES). Let \( W \) be a 2-dimensional \( \mathbb{C} \)-vector space, \( G = \text{SL}(2, \mathbb{C}) \), and \( V_n := \text{Sym}^n(W) \) which can be identified as a vector space of homogeneous polynomials of degree \( n \) on \( V \). Let \( \mathbb{P}(V_n) \) be the space of 0-cycles of \( n \) unordered points on the projective line \( \mathbb{P}(W) \), the zeros of \( f \in V_n \) determining the cycle.

If \( f = \sum_{i=0}^{n} a_i x^{n-i} y^i \) and \( \lambda \) is the one parameter subgroup given by

\[
t \to \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}
\]

in these coordinates \( x, y \), then

\[
\lambda(t)f = \sum_{i=0}^{n} a_i (tx)^{n-i}(t^{-1}y)^i = \sum_{i=0}^{n} a_i t^{n-2i} x^{n-i} y^i.
\]

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The main references of this note are [1] and [2].
The set of weights of $f$ with respect to this $\lambda$ is \( \{ n - 2i | a_i \neq 0 \} \). For $f$ to be stable, this set of weights has both positive and negative values which holds only if there exist $k < n/2$ and $\ell > n/2$ such that $a_k \neq 0$ and $a_\ell \neq 0$, i.e. the non-zero coefficients of $f$ must lie on both side of 0.

\[
\begin{array}{cccccc}
a_n & a_{n-1} & a_1 & a_0 & \text{coefficient} \\
-n & -n + 2 & 0 & n - 2 & n & \text{weight}
\end{array}
\]

This condition is equivalent to that if $j \geq n/2$, neither $x^j$ nor $y^j$ divide $f$. One can check directly that the stability of $f$ is equivalent to the same condition with respect to all linear forms $l$: $l \nmid f$ if $j \geq n/2$. This condition is equivalent to that $f$ has no zero of multiplicity greater or equal to $n/2$.

Let $\mathbb{P}(V_n)_s$ (resp. $\mathbb{P}(V_n)_{ss}$) be the set of stable (resp. semi-stable) 0-cycles in $P(V_n)$. Then

\[
\begin{align*}
\mathbb{P}(V_n)_s &= \{ 0 - \text{cycles with no points of multiplicity} \geq n/2 \} \\
\mathbb{P}(V_n)_{ss} &= \{ 0 - \text{cycles with no points of multiplicity} > n/2 \}.
\end{align*}
\]

**Example (CURVES).** Let $W$ be a 3-dimensional $\mathbb{C}$-vector space, $G = \text{SL}(3, \mathbb{C})$, and $V_n := \text{Sym}^n(W) = \text{as before}$. Then a point $f = \sum_{x^i y^j z^k} \alpha_{i,j,k} x^i y^j z^k \in V_n$ defines a plane curve of degree $n$. Let $\lambda$ be the one parameter subgroup given by

\[
t \to \begin{pmatrix} t^a & 0 \\ 0 & t^b \\ \end{pmatrix}
\]

in these coordinates $x, y, z$ where $a + b + c = 0$.

Then $\lambda(t)f = \sum_{x^i y^j z^k} \alpha_{i,j,k} t^{ai_x + bi_y + ci_z} x^i y^j z^k$, and the corresponding weights are

\[
\{ ai_x + bi_y + ci_z | \alpha_{i,j,k} \neq 0, a + b + c = 0, i_x + i_y + i_z = n \}.
\]
Let \( L \) be the line defined by \( a_i x + b_j y + c_k z = 0 \), \((a, b, c) \neq 0\), which should pass \((1, 1, 1)\) since \(a + b + c = 0\). Consider the following triangle representing \( f \)

\[
\begin{align*}
&x^n \\
&x^{n-1}y \\
&x^{n-2}y^2 \\
&\vdots \\
&x^n y^n \\
&y^{n-1}z \\
&\vdots \\
&y z^{n-1} \\
&z^n
\end{align*}
\]

We can conclude:

\( f \) is unstable \(\iff\) in some coordinates, all non-zero coefficients of \( f \) lie to one side of some \( L \).

\( f \) is semi-stable \(\iff\) for all choices of coordinates, \( f \) has non-zero coefficients on both sides of \( L \) or on \( L \).

\( f \) is stable \(\iff\) for all choices of coordinates, \( f \) has non-zero coefficients on both sides of \( L \).
$n = 2$: For a nonsingular quadric $f$, we may choose coordinates so that $(1, 0, 0) \in f$ and $z = 0$ is the tangent line at $(1, 0, 0)$. Therefore, the coefficient of $x^2$ in $f$ is zero, and the coefficient of $xz$ is not zero. Changing coordinates again, we may assume that the coefficient of $xy$ is zero. Therefore we have the following diagram:

This shows that $f$ is semi-stable, but not stable.

For a singular quadric, we may assume $(1, 0, 0) \in f$ and it is a singular point. Then it is easy to see that the coefficients of $x^2$, $xy$, and $xz$ are all zero. Therefore, we have the following diagram:

This shows that $f$ is unstable.
$n = 3$: Let $f$ be degree 3 homogeneous polynomial in three variables $x, y, z$. We may choose coordinates so that $f(1, 0, 0) = 0$. (Note, for elliptic curves $E$ we usually choose $(0, 0, 1) \in E$. Therefore, one needs to adjust definitions of all invariants. For example, under our context, an elliptic curve may be represented by $f = xy^2 - z^3 + Ax^2z + Bx^3$ with $\Delta = 27A^3 + B^2 \neq 0$.) In general, one can define $A$ and $B$ in terms of coefficients of $f$ with degree 4 and 6 respectively, we refer to [4] for definition. Then up to a constant the $j$-invariant is $A^3/\Delta$. We have the following table to include all possibilities

<table>
<thead>
<tr>
<th>SINGULARITIES OF $f$</th>
<th>“WORST” TRIANGLE</th>
<th>STABILITY AND INDEPENDENTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f$ has a triple point</td>
<td><img src="image1" alt="Triangular Diagram" /></td>
<td>Unstable $\Delta = 0$, but $A, B \neq 0$</td>
</tr>
<tr>
<td>$f$ has a cusp or two components tangent at a point</td>
<td><img src="image2" alt="Triangular Diagram" /></td>
<td>unstable $A = B = 0$</td>
</tr>
<tr>
<td>$f$ has ordinary double points (including the reducible cases: $f$ is a conic and a transversal line, or $f$ is a triangle)</td>
<td><img src="image3" alt="Triangular Diagram" /></td>
<td>semi-stable and not stable $\Delta = 0$, but $A, B \neq 0$, $j = \infty$</td>
</tr>
<tr>
<td>$f$ is smooth</td>
<td><img src="image4" alt="Triangular Diagram" /></td>
<td>stable $\Delta \neq 0$ $j$ finite</td>
</tr>
</tbody>
</table>
Let $H$ be the upper half-plane of $\mathbb{C}$, $H^* = H \cup \mathbb{Q} \cup \{\infty\}$, the extended upper half-plane, $\Gamma(1) = \text{SL}(2, \mathbb{Z})/\{\pm 1\}$, $Y(1) = \Gamma(1) \setminus H$ and $X(1) := \Gamma(1) \setminus H^*$ be the modular curve. Recall that each $\tau \in H$ associated a lattice $\Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z}$ and an elliptic curve $\mathbb{C}/\Lambda_\tau$. The $j$-invariant function gives a bijection

$$Y(1) \to \mathbb{C}, \quad \tau \to j(\mathbb{C}/\Lambda_\tau).$$

Indeed, it gives a complex analytic isomorphism $j : X(1) \sim \to \mathbb{P}^1(\mathbb{C})$. $Y(1)$ is correspond to the set of stable points, and $X(1)$ is correspond to the set of semi-stable points.

2. General varieties

In general, a projective variety may not be defined by a single equation. Therefore, we need to introduce Chow forms to define stability.

Chow Forms. Let $V$ be a closed subvariety of dimension $n$ in $\mathbb{P}^N$. Let $\hat{\mathbb{P}}^N$ denote the projective space dual to $\mathbb{P}^N$ and identify points of $\hat{\mathbb{P}}^N$ with hyperplanes in $\mathbb{P}^N$.

On can show that the set

$$Y_V = \{(H_0, \ldots, H_n) \in (\hat{\mathbb{P}}^N)^{n+1} | V \cap H_0 \cap \cdots \cap H_n \neq \emptyset\},$$

is a hypersurface in $(\hat{\mathbb{P}}^N)^{n+1}$. The multihomogeneous form $F_V$ that defines $Y_V$ is unique up to a scalar and is called the Chow form of $V$. Furthermore, if $V'$ is another subvariety of the same dimension, then $F_{V'}$ is a scalar multiple of $F_V$ if and only if $V = V'$.

Chow Stability. A projective variety $V \subset \mathbb{P}^N$ is Chow stable or simply stable if its Chow form is stable for the natural $\text{SL}(N+1)$-action.

Notation. If $p(a)$ is an integer-valued function which is represented by polynomial over $\mathbb{Q}$ of degree at most $n$ in $m$ for large $m$, we will denote by n.l.c.($p$) the integer $e$ for which

$$p(m) = e \frac{m^n}{n!} + \text{lower order terms}.$$

Proposition 1 (Hilbert-Samuel Polynomial), Suppose that $X^n$ is a k-variety, $L$ is an invertible sheaf on $X$ and $\mathcal{I} \subset \mathcal{O}_X$ is an ideal sheaf such that $Z = \text{Supp} \mathcal{O}_X/\mathcal{I}$ is proper over $k$. Then there is a polynomial $P(r, m)$ of total degree $\leq n$ such that for large $m$

$$\chi(L^r/\mathcal{I}^mL^r) = P(r, m).$$

Definition. We denote by $e_L(\mathcal{I})$ (the multiplicity of $\mathcal{I}$ measured via $L$) the integer n.l.c.($\chi(L^r/\mathcal{I}^mL^r)$).

Example. (i) If $\mathcal{I} = 0$ and $X$ is complete, $P$ is the Hilbert polynomial of $L$.

(ii) If $Z$ is set theoretically a point $x$, then $P$ is the Hilbert-Samuel polynomial of $\mathcal{I}$ as an ideal of $\mathcal{O}_{x,X}$ and $e(\mathcal{I})$ is its multiplicity there. In particular, it is independent of $L$. 

2.1. Classical Geometric Interpretation. Let $X^n \subset \mathbb{P}^N$ be a projective variety. $L = \mathcal{O}_X(1)$, and $\Lambda$ be a subspace of $\Gamma(\mathbb{P}^N, \mathcal{O}_X(1))$. Define $L_\Lambda$ to be the linear subspace of $\mathbb{P}^n$ given by $s = 0$, $s \in \Lambda$. Define $\mathcal{I}_\Lambda$ to be the ideal sheaf generated by sections $s \in \Lambda$, i.e. $\mathcal{I}_\Lambda \cdot L$ is the subsheaf of $L$ generated by those sections and $Z = \text{Supp} \mathcal{O}_X/\mathcal{I}_\Lambda = X \cap L_\Lambda$ is the set of their base points.

Example. Let $X^1 \subset \mathbb{P}^2$ be a curve of degree $d$, and $\Lambda = \mathbb{C}X_0 + \mathbb{C}X_1 \subset \Gamma(\mathbb{P}^2, \mathcal{O}(1))$. Then $L_\Lambda = \{X_0 = X_1 = 0\} \subset \mathbb{P}^2$, $\mathcal{I}_\Lambda$ is the ideal sheaf generated by the sections $s \in \Lambda$ and $Z = \text{Supp} \mathcal{O}_X/\mathcal{I}_\Lambda = \{y = (0, 0, 1)\}$. Let $p_\Lambda(x_0, x_1, x_2) = (x_0, x_1)$. The picture is:

Then $p_\Lambda(X) = (a\mathbb{P}^1)$ where $a$ is the degree of the covering $p_\Lambda$; a generic line meets $X$ in $d$ points and as this line specializes to a non-tangent line through $y$ meets $X$ at $y$ with mutiplicity equal to $e_L(\mathcal{I}_\Lambda)$ and meets $X$ away from $y$ in $d - e_L(\mathcal{I}_\Lambda) = a$ points.

Back to the general situation. If $p_\Lambda : \mathbb{P}^N - L_\Lambda \rightarrow \mathbb{P}(\Lambda) = \mathbb{P}^m$ is the canonical projection and $\pi$ is the blow-up of $X$ along $\mathcal{I}_\Lambda$ then there is a unique map $q$ making the following diagram commute:

Moreover, because sections of $\mathcal{O}_{\mathbb{P}^m}(1)$ pull back to sections of $\mathcal{I}_\Lambda \cdot L$ on $X$ and are blown-up to sections of $L$ twisted by minus the exceptional divisor $E$,

$$q^*(\mathcal{O}_{\mathbb{P}^m}(1)) = (\pi^*L)(-E).$$
Define \( p_\Lambda(X) \), the image of \( X \) by the projection \( p_\Lambda \), to be \( q_\ast(B) \), i.e. \( q(B) \) with multiplicity equal to the degree of \( B \) over \( q(B) \) if these have the same dimension and 0 otherwise.

The following is the geometric interpretation of \( e_L(\mathcal{I}_\Lambda) \).

**Proposition 2.** \( e_L(\mathcal{I}_\Lambda) = \deg X - p_\Lambda(X) \).

**Proof.** Let \( H \) be the divisor class of a hyperplane section of \( X \), then

\[
\deg X = H^n = \text{n.l.c.}(\chi(O_X(m))).
\]

By [1], \( q \) is defined by the linear system of divisors of the form \( \pi^{-1}(H) - E \), hence

\[
\deg p_\Lambda(x) = (\pi^{-1}(H) - E)^n = \text{n.l.c.}(\chi(O_X(m)(-mE))) = \text{n.l.c.}(\mathcal{I}_\Lambda O_X(m)).
\]

From its definition

\[
e_L(\mathcal{I}_\Lambda) = \text{n.l.c.}(\chi(O_X(m)/\mathcal{I}_\Lambda O_X(m)))
= \text{n.l.c.}(\chi(O_X(m))) - \text{n.l.c.}(\mathcal{I}_\Lambda O_X(m))
= \deg X - \deg p_\Lambda(X).
\]

\[\square\]

2.2. Stability Criteria Involving Degree of Contact with Weights. Fix a projective variety \( X^n \subset \mathbb{P}^N \), coordinates \( X_0, \ldots, X_N \) on \( \mathbb{P}^n \) and a 1-parameter subgroup \( \lambda \) of \( \text{SL}(N+1) \) which in these coordinates is given by

\[
\lambda(t) = \begin{bmatrix}
t^{r_0} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & t^{r_N}
\end{bmatrix}
\]

where \( r_0 \geq r_1 \geq \cdots \geq r_N = 0 \) and \( k = \frac{\sum_{i=0}^{N} r_i}{N+1} \). Consider the following filtration:

\[
H^0(X, O_X(1)) = V_0 \supset V_1 \supset \cdots \supset V_N,
\]

with \( V_i = \text{span}\{X_i, \ldots, X_N\} \) in \( H^0(X, O_X(1)) \). We call the choice of such a filtration with weights a weighted flag in \( H^0(X, O_X(1)) \).

Let \( \tilde{X} = X \times \mathbb{A}^1, \ O_{\tilde{X}}(1) = O_X(1) \otimes O_{\mathbb{A}^1}, \) and \( t \) be the coordinate on \( \mathbb{A}^1 \). To a weighted flag \( \mathcal{F} \), we associate the \( \mathbb{C}[t] \) submodule \( I_\mathcal{F} \) of \( \Gamma(\tilde{X}, O_{\tilde{X}}(1)) \) generated by \( \{t^{r_0}X_0, t^{r_1}X_1, \ldots, t^{r_N}X_N\} \) and the ideal sheaf \( \mathcal{I}_\mathcal{F} \subset O_{\tilde{X}} \) which is defined by

\[
\mathcal{I}_\mathcal{F} \cdot O_{\tilde{X}}(1) = \text{sheaf generated by } I_\mathcal{F} \text{ in } O_{\tilde{X}}(1).
\]

**Remark.** \( \mathcal{I}_\mathcal{F} \) is supported only over \( 0 \in \mathbb{A}^1 \), and over the hyperplane section \( x_N = 0 \).

**Notation.** We denote by \( e_\mathcal{F} \) the multiplicity \( e_{O_{\tilde{X}}(1)}(\mathcal{I}_\mathcal{F}) \).

We say that \( (X, O_X(1)) \) is stable (resp. semistable, unstable) with respect to \( \lambda \) if its Chow form is stable (resp. semistable, unstable) with respect to \( \lambda \).

Let \( F_X(u_0, \ldots, u_n) \) be the Chow form of \( X \) with respect to the embedding given by \( X_0, \ldots, X_N \). It is a multihomogeneous polynomial of degree \( \deg(X) \) in each set of the variables \( u_i = (u_{i0}, \ldots, u_{iN}), \ i = 0, \ldots, n \). Let \( t \) be an auxiliary variable, and
\[ c = (c_0, ..., c_N) \] where \( c_i \)'s are integers and \( c_0 \geq c_1 \cdots \geq c_N \). We consider the decomposition

\[
F_X(t^{c_0}u_{00}, ..., t^{c_N}u_{0N}, ..., t^{c_N}u_{nN}) = \sum_j t^{\rho_j} F_j(u_0, ..., u_n),
\]

where \( F_j(u_0, ..., u_n) \) are polynomials not containing the variable \( t \). Let \( c_i = r_i - k, \) \( 0 \leq i \leq N \). By definition, the Chow form \( F(u_0, ..., u_n) \) is stable (resp. semi-stable) with respect to \( c \) (or say, \( \lambda \)) if and only if

\[
\min\{\rho_j : F_j \neq 0\} < 0 \quad \text{(resp.} \leq 0)\text{).}
\]

For simplicity, we say \( \rho_F(c) = \min\{\rho_j : F_j \neq 0\} \) is the \( c \)-weight of the Chow form \( F_X \). It is easy to see the following.

**Proposition 3.** (1). Let \( r = (k + c_0, ..., k + c_N) \), then

\[
\rho_F(c) = \rho_F(r) - k \cdot (n + 1) \deg X.
\]

(2). Let \(-c = (-c_0, ..., -c_N)\), and let

\[
F_X(t^{c_0}u_{00}, ..., t^{c_N}u_{0N}, ..., t^{c_N}X_{nN}) = \sum_j t^{\rho_j} F_j(u_0, ..., u_n),
\]

then

\[
\rho_F(-c) = -\max\{\rho_j : F_j \neq 0\}.
\]

**Remark.** \( \max\{\rho_j : F_j \neq 0\} \) is the Chow weight of \( X \) with respect to \( c \).

The given 1-parameter subgroup \( \lambda(t) \) is associated with an integer vector \( r = (r_0, ..., r_N) \). Mumford showed that the \( r \)-weight of the Chow form \( F_X \) equals the degree of contact \( e_{\mathcal{O}_X(1)}(\mathcal{F}_X)(= e_{\mathcal{F}_X}, \text{for short}) \). Therefore, we can deduce the following.

**Theorem 4** (Mumford). Fix \( X^n \subset \mathbb{P}^N \) a projective variety and a 1-parameter subgroup \( \lambda \) given as in (*) with \( k = (\sum_{i=0}^N r_i)/N + 1 \). Then \( (X, \mathcal{O}_X(1)) \) is stable (resp. semistable) with respect to \( \lambda \) if and only if

\[
e_{\mathcal{F}_X} < \text{(resp.} \leq \frac{(n + 1) \deg X}{N + 1} \sum_{i=0}^N r_i\text{).}
\]

### 2.3. Computing Degree of Contact with Weights.

The situation is still the same as in the last subsection. We will relate the multiplicity \( e_{\mathcal{F}_X} \) with certain weight of a Hilbert polynomial.

Let \( r = (r_0, ..., r_N) \) be the \( N + 1 \)-tuple of positive reals associated with the 1-parameter subgroup \( \lambda \). We define the weight of \( X_i \) to be \( r_i \), the weight of a monomial in the \( X_i \) to be the sum of the weights of the \( X_i \) occurring in it, and the weight of a polynomial in the \( X_i \)'s to be the greatest weight of a monomial occurring in it. When \( m \) is large enough, the monomials of degree \( m \) in the \( X_i \)'s generate \( H^0(X, \mathcal{O}_X(m)) \) by \([3]\). We then define the weight of an element of \( H^0(X, \mathcal{O}_X(m)) \) to be the least weight of the polynomials in the \( X_i \)'s reducing to it. The weight of a basis of \( H^0(X, \mathcal{O}_X(m)) \) is the sum of the weights of its members. We often consider monomials in the \( X_i \)'s as elements of \( H^0(X, \mathcal{O}_X(m)) \) by abuse of language. A basis consisting of such monomials is called an \( \lambda \)-basis, and the minimum weight of such basis will be denoted \( w_{\lambda}(m) \).
Let $I = I_X$ be the homogeneous ideal of $\mathbb{C}[X_0, \ldots, X_N]$ defining $X$. Let $m$ be a positive integer, denote by $\mathbb{C}[X_0, \ldots, X_N]_m$ the homogeneous polynomial of degree $m$ and $I_m := \mathbb{C}[X_0, \ldots, X_N]_m \cap I$. Then we can identify $H^0(X, \mathcal{O}_X(m)) = \Gamma(X, \mathcal{O}_X(m))$ to be $\mathbb{C}[X_0, \ldots, X_N]_m/I_m$. Moreover, \[
\dim H_X(m) := \dim (\mathbb{C}[X_0, \ldots, X_N]_m/I_m)\]
is called the $m$-th Hilbert polynomial of $X$. Under this identification, it is easy to see that \[
 w_r(m) = \min \left( \sum_{i=1}^{H_1(m)} a_i \cdot r \right), \]
where the minimum is taken over all sets of monomials $x^{a_1}, \ldots, x^{a_{H_1(m)}}$ whose residue classes modulo $I$ form a basis of $\mathbb{C}[x_0, \ldots, x_N]_m/I_m$.

**Remark.** The Hilbert weight of $X$ with respect to $r$ is defined by \[
 S_X(m, r) := \max \left( \sum_{i=1}^{H_1(m)} a_i \cdot r \right), \]
where the maximum is taken over all sets of monomials $x^{a_1}, \ldots, x^{a_{H_1(m)}}$ whose residue classes modulo $I$ form a basis of $\mathbb{C}[x_0, \ldots, x_N]_m/I_m$. Let $-r = (-r_0, \ldots, -r_N)$. Then \[
 w_{-r}(m) = -S_X(m, r). \]

**Proposition 5.** Fix $X$, $\mathcal{F}$, $I_\mathcal{F}$ and $\mathcal{J}_\mathcal{F}$ as in §2.2. Then for large $m$, \[
 \dim(\Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}(m)/I_{\mathcal{F}}^{\otimes m})) = w_r(m). \]

**Proof.** Let $W_k \subset \Gamma(X, \mathcal{O}_X(m))$ be the $k$-th graded piece of $t$ so that $W_0 \subset W_1 \cdots \subset W_M = \Gamma(X, \mathcal{O}_X(m))$ for $M$ large. Since $W_j$ is spanned by the monomials in $X_i$ of weight less than or equal to $k$ when $m$ is large, there is an $\mathcal{F}$-basis of $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(m))$ compatible with this filtration. Such a basis clearly has minimal weight and hence \[
 w_r(m) = 0 \cdot \dim W_0 + 1 \cdot (\dim W_1 - \dim W_0) + \cdots + j(\dim W_j - \dim W_{j-1}) + \cdots + M(\dim W_M - \dim W_{M-1}) = M \dim \Gamma(X, \mathcal{O}_X(m)) - \dim W_0 - \cdots - \dim W_{M-1} = \sum_{j=0}^{M-1} \dim(\Gamma(X, \mathcal{O}_X(m))/W_j) = \dim(\Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}(m)/I_{\mathcal{F}}^{\otimes m})). \]

We note that the last equality comes from the following identification:
\[
 I_{\mathcal{F}} = \sum_{j=0}^{M} t^j W_j, \]
and \[
 \Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}(m)) = \Gamma(X, \mathcal{O}_X(m)) \otimes k[t]. \]

The relation between $e_\mathcal{F}$ and $w_r(m)$ is as follows.

**Corollary 6.** $e_\mathcal{F} = \text{n.l.c.}(w_r(m))$. 

Proof. 
\[
\begin{align*}
\epsilon_F &= \epsilon_{O_X}(1)(\mathcal{F}) \\
&= \text{n.l.c.}(\chi(O_X(m)/\mathcal{F}^m)) \\
&= \text{n.l.c.}(\dim(\Gamma(\tilde{X}, O_{\tilde{X}}(m)/I_F^{\otimes m}))) \\
&= \text{n.l.c.}(w_F(m)).
\end{align*}
\]

\[\square\]

The following example from [1] illustrates the application of Theorem 4.

**Example.** The Steiner surface \(X^2 \subset \mathbb{P}^4\) is defined to be the closure of the image of the map \(\Psi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^4\) defined by
\[
(x, y, z) \rightarrow (xz, yz, x^2, xy, y^2) = (w_0, ..., w_4).
\]
\(\Psi\) is a rational map undefined only at \(P = (0, 0, 1)\). Then \(\Psi\) extends to a morphism from the blowing-up of \(\mathbb{P}^2\) at \(P\) which is a rational rule surface of type \(\mathbb{F}_1\) ruled by the pencil of lines through \(P\). Therefore, \(X\) is just \(\mathbb{P}^2\) blown up at the point \(P\) and embedded by the system of conics passing through \(P\). The degree of \(X\) is three. This can be checked for example, counting intersection of \(X \cap \{w_0 - w_1 = 0\} \cap \{w_3 = 0\}\) which consists of three points \((0, 0, 1, 0, 0), (0, 0, 0, 0, 1)\) and \((1, 1, 0, 0, 0)\) of multiplicity one.

Let \(\mathcal{F}\) be the weighted flag given by choosing coordinates \(w_i\) as above with weights \(r_0 = r_1 = 1\) and \(r_2 = r_3 = r_4 = 0\). Since the exceptional divisor is defined by \(w_2 = w_3 = w_4 = 0\), \(\mathcal{F}\) measures the vanishing of sections on that curve. The 1-parameter subgroup corresponding to it is
\[
\lambda(t) = \begin{bmatrix} t & 0 \\ 0 & 1 \\ t^{-\frac{2}{5}} & 0 \\ 1 & 1 \end{bmatrix}.
\]
So we get,
\[
\frac{(\dim X + 1)(\deg X)}{N + 1} \sum_{i=0}^{N} r_i = \frac{3 \cdot 3 \cdot 2}{5} = \frac{18}{5}.
\]

We will now compute \(e_F\) by Proposition 5 and Corollary 6. Let \(i, j, k\) and \(\ell\) be non-negative integers. Then we can write
\[
\Gamma(X, O_X(m)) = \text{span}\{x^i y^j z^k | i + j + k = 2m, k \leq m\},
\]
\[
\Gamma(\tilde{X}, O_{\tilde{X}}(m)) = \text{span}\{x^i y^j z^k t^\ell | i + j + k = 2m, k \leq m\},
\]
\[
I_F = \text{span}\{x^i y^j z^k t^\ell | i + j + k = 2, k \leq 1, \ell \geq k\},
\]
\[
I_F^{\otimes m} = \text{span}\{x^i y^j z^k t^\ell | i + j + k = 2m, \ell \geq k \leq m\}.
\]
Hence
\[\dim \frac{\Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}(m))}{I_{\tilde{X}}^m} = \sum_{k=0}^{m} \# \{(i, j, \ell) | i + j = 2m - k, \ell < k\}\]
\[= \sum_{k=0}^{m} (2m - k + 1)k = 2m \sum_{k=0}^{m} k - \sum_{k=0}^{m} k^2 + \sum_{k=0}^{m} k = \frac{2}{3} m^3 + O(m^2) = 4 \left(\frac{m^3}{3!}\right) + O(m^2).\]

This shows that \(e_{\mathcal{F}} = 4\) by Proposition 5 and Corollary 6.

Together with (2), we have
\[e_{\mathcal{F}} > \frac{(\dim X + 1)(\deg X)}{N + 1} \sum_{i=0}^{N} r_i.\]

Hence, \(X\) is Chow-unstable with respect to \(\mathcal{F}\) by Theorem 4.

References