Singular integral operators on Triebel–Lizorkin spaces of para-accretive type

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\begin{abstract}
In this article, we study the boundedness of singular integral operators acting on Triebel–Lizorkin spaces of para-accretive type and show a $T_b$ theorem on these spaces, which extends previous results in David, Journé and Semmes (1985) [3], Han (1994) [9], Han, Lee and Lin (2004) [10], Han and Sawyer (1990) [12], Lin and Wang (2009) [15], Wang (1999) [22].
\end{abstract}

\section{Introduction}
Since Calderón and Zygmund developed the theory of singular integral operators in the 1950's, there have been lots of enthusiasm to generalize the theory in various ways. One interest is to consider the boundedness of operators on Hardy spaces, Triebel–Lizorkin spaces or Besov spaces (cf. [5,8,10–12,15–17,19,21,22]). The other interests include considering non-convolution type kernel (e.g. the $T_1$ and $T_b$ theorems [2,3]) or investigating operator-valued kernels (cf. [13,14,18]).

Frazier, Torres, and Weiss [8] considered the $T_1$ theorem on Triebel–Lizorkin spaces $\dot{F}^{p,q}_\alpha$, which include the classical $L^p$ spaces for $1 < p < \infty$ and Hardy spaces $H^p$ for $0 < p \leq 1$, under the assumption $T\chi^\gamma = T^*\chi^\gamma \equiv 0$ for a certain condition on $\gamma$. Several years later, the second named author [21,22] of the current paper extended the boundedness of singular integral operators acting on $\dot{F}^{p,q}_\alpha$ to more relaxed restriction on $T\chi^\gamma$ and $T^*\chi^\gamma$.

If function 1 in the $T_1$ theorem is replaced by an accretive function, a bounded complex-valued function $b$ satisfying $0 < \delta \leq \text{Re} b(x)$ almost everywhere, McIntosh and Meyer [17] showed the $L^2$ boundedness of the Cauchy integral on all Lipschitz curves. David, Journé, and Semmes [3] gave more general conditions on $L^\infty$ functions, so called para-accretive functions, and proved a new $T_b$ theorem by substituting function 1 for para-accretive functions. It was also shown in [3] that if $T_b$ theorem holds for a bounded function $b$, then $b$ is necessarily para-accretive.

Recently, the authors [15] used a discrete Calderón-type reproducing formula and Plancherel–Pôlya-type inequality to characterize homogeneous Triebel–Lizorkin spaces of para-accretive type $\dot{F}^{p,q}_{b,p}$. A necessary and sufficient condition of singular integral operators which is bounded from $\dot{F}^{0,q}_{1,p}$ to $\dot{F}^{0,q}_{b,p}$ with the regularity exponent $\epsilon$

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of the kernel, is also derived in [15]. In this article, we study the $F_{1,p}^{0,q} - F_{b,p}^{0,q}$ boundedness of singular integral operators for wider ranges of $\frac{n}{n+\varepsilon} < p < \infty$ and $\frac{n}{n+\varepsilon} < q \leq \infty$.

We begin by recalling some basic results about Calderón–Zygmund operator theory. As usual, we denote by $\mathcal{D}$ the set of $C^\infty$ functions with compact support.

**Definition 1.1.** We say that $T$ is a **singular integral operator**, denoted by $T \in \text{SIO}(\varepsilon)$, if $T$ is a continuous linear operator from $\mathcal{D}(\mathbb{R}^n)$ into its dual associated to a kernel $K(x, y)$, a continuous function defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{x = y\}$, satisfying the following conditions: there exist constants $C > 0$ and $0 < \varepsilon \leq 1$ such that

$$|K(x, y)| \leq C \frac{1}{|x - y|^n} \quad \text{for all } x \neq y;$$

$$|K(x, y) - K(x', y)| \leq C \frac{|x - x'|^\varepsilon}{|x - y|^{n+\varepsilon}} \quad \text{for all } x, x', y \in \mathbb{R}^n \text{ with } |x - x'| \leq \frac{|x - y|}{2};$$

$$|K(x, y) - K(x, y')| \leq C \frac{|y - y'|^\varepsilon}{|x - y'|^{n+\varepsilon}} \quad \text{for all } y, y', x \in \mathbb{R}^n \text{ with } |y - y'| \leq \frac{|x - y|}{2}. $$

Moreover, the operator $T$ can be represented by

$$\langle Tf, g \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y) f(y) g(x) \, dy \, dx$$

for all $f, g \in \mathcal{D}(\mathbb{R}^n)$ with $\text{supp}(f) \cap \text{supp}(g) = \emptyset$.

We say that a singular integral operator is a **Calderón–Zygmund operator** if it can be extended to a bounded operator on $L^2(\mathbb{R}^n)$. Coifman and Meyer [1] showed that every Calderón–Zygmund operator is bounded on $L^p$ for $1 < p < \infty$.

Let $C^\eta_0$ denote the space of continuous functions $f$ with compact support such that

$$\|f\|_\eta := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\eta} < \infty.$$ 

A locally integrable function defined on $\mathbb{R}^n$ belongs to $\text{BMO}$ if it satisfies

$$\|f\|_{\text{BMO}} := \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx < \infty,$$

where the supremum is taken over all cubes $Q \subseteq \mathbb{R}^n$ whose sides are parallel to the axes and $f_Q = \frac{1}{|Q|} \int_Q f(x) \, dx$. Note that these cubes need not be dyadic.

**Definition 1.2.** Let $T : C^\eta_0 \rightarrow (C^\eta_0)', \eta > 0$, be a continuous linear operator. We say that $T$ has the **weak boundedness property**, denoted by $T \in \text{WBP}$, if, for each $\eta > 0$, there is a constant $C > 0$ such that, for all cubes $Q$ with diameter at most $t > 0$ and all $f, g \in C^\eta_0$ supported in $Q$,

$$\|Tf, g\| \leq C t^{n+2\eta} \|f\|_\eta \|g\|_\eta.$$ 

David and Journé [2] gave a general criterion for the $L^2$ boundedness of singular integral operators as follows.

**Proposition 1.3** ($T^1$ theorem for $L^2$). Suppose that $T$ is a singular integral operator and $T^*$ denotes its transpose. Then $T$ is bounded on $L^2$ if and only if

(i) $T1 \in \text{BMO}$,

(ii) $T^*1 \in \text{BMO}$, and

(iii) $T \in \text{WBP}$.

Before stating the $Tb$ theorem of David, Journé, and Semmes [3], we recall some definitions.

**Definition 1.4.** A bounded complex-valued function $b$ defined on $\mathbb{R}^n$ is said to be **para-accretive** if there exist constants $C, \gamma > 0$ such that, for all cubes $Q \subseteq \mathbb{R}^n$, there is a subcube $Q' \subseteq Q$ with $\gamma |Q| \leq |Q'|$ satisfying

$$\left| \frac{1}{|Q'|} \int_{Q'} b(x) \, dx \right| \geq C > 0.$$
**Definition 1.5.** Suppose $b_1$ and $b_2$ are bounded complex-valued functions whose inverses are also bounded. A generalized singular integral operator is a continuous linear operator $T$ from $b_1 C_0^0$ into $(b_2 C_0^0)'$, $\eta > 0$, for which the associated kernel $K(x, y)$ satisfies inequalities (1)-(3) such that, for all $f, g \in C_0^0$ with $\text{supp}(f) \cap \text{supp}(g) = \emptyset$,

$$
\langle T b_1 f, b_2 g \rangle = \int_\mathbb{R} \int_\mathbb{R} g(x) b_2(x) K(x, y) b_1(y) f(y) \, dx \, dy.
$$

Such an operator $T$ is written as $T \in \text{GSIO}(\epsilon)$, where $\epsilon$ is the regularity exponent of $K(x, y)$ in Definition 1.1.

Denote $M_b$ the multiplication operator by $b$; that is, $M_b f = bf$. David, Journé and Semmes [3] proved the following $Tb$ theorem.

**Proposition 1.6** ($Tb$ theorem for $L^2$). Suppose that $b_1$ and $b_2$ are para-accretive functions and $T \in \text{GSIO}(\epsilon)$. Then $T$ is bounded on $L^2$ if and only if

(i) $T b_1 \in \text{BMO}$,
(ii) $T b_2 \in \text{BMO}$, and
(iii) $M_{b_2} T M_{b_1} \in \text{BWP}$.

The main purpose of this paper is to demonstrate a $Tb$ theorem in Triebel–Lizorkin spaces of para-accretive type $\dot{F}^{0,q}_{b,p}$, which was introduced by Han [9] for $p, q > 1$, by Deng and Yang [4] for $p, q \leq 1$, denoted as $b^{-1} \dot{F}^{0,q}_{p,q}$, and by Lin and Wang [15] for larger ranges of $p$ and $q$. Here is our main result in this article.

**Theorem 1.7** ($Tb$ theorem for Triebel–Lizorkin spaces). Let $b$ be a para-accretive function, $T \in \text{GSIO}(\epsilon)$ and $M_b T M_b \in \text{BWP}$ for some $\epsilon > J^*$, where $J = n/\min\{1, p, q\}$, $J^* = J - \lfloor J \rfloor$, and $\lfloor \cdot \rfloor$ denotes the integer function. Write $U_b = \bigcup_{0 < \delta \leq \epsilon} K_b^{\delta,\infty}$. Then $T$ extends to a bounded linear operator from $\dot{F}^{0,q}_{1,p}$ to $\dot{F}^{0,q}_{b,p}$ if one of the following cases holds:

(a) $q = 2$, $1 < p < +\infty$, $T b \in \dot{F}^{0,2}_{b,\infty}$, and $T^* b \in \dot{F}^{0,2}_{b,\infty}$;
(b) $1 < q < 2$, $1 < p < +\infty$, $T b \in \dot{F}^{0,q}_{b,\infty}$, and $T^* b \in U_b$;
(c) $2 < q < 2$, $1 < p < +\infty$, $T b \in \dot{F}^{0,q}_{b,\infty}$, and $T^* b \in \dot{F}^{0,q'}_{b,\infty}$;
(d) $n + \epsilon < q < 1, 1 < p < +\infty$, $T b \in \dot{F}^{0,q}_{b,\infty}$, and $T^* b = 0$;
(e) $n + \epsilon < q < 2, n + \epsilon < p < 1$, $T b \in \dot{F}^{0,q}_{b,\infty}$, and $T^* b = 0$;
(f) $2 < q < +\infty, n + \epsilon < p < 1$, $T b \in U_b$, and $T^* b = 0$;
(g) $2 < q < +\infty, p = +\infty, T b = 0$, and $T^* b \in \dot{F}^{0,q'}_{b,\infty}$;
(h) $n + \epsilon < q < 2, p = +\infty, T b = 0$, and $T^* b \in U_b$.

**Remark 1.8.** By the Calderón–Zygmund operator theory, any Calderón–Zygmund operator $T$ is also bounded on $L^p$ for all $1 < p < \infty$, bounded from $H^p$ to $L^p$ for $\frac{n}{n+p} < p \leq 1$, and bounded from $H^p$ to itself for $\frac{n}{n+p} < p \leq 1$ if $T^*(1) = 0$. However, in general, $T$ is not bounded on $H^p$ if $T^*(b) = 0$, where $b$ is a para-accretive function.

Part (a) is David, Journé and Semmes’ result, which is included here for comparison. From part (a), it exhibits a particular role played by the index $q = 2$ (see Lemma 3.2). Note that case (e) includes $H^p_b \equiv \dot{F}^{0,2}_{b,\infty}$ for $\frac{n}{n+p} < p \leq 1$. By Theorem 3.6, the theorem is sharp. For example, under the conditions $T^* b = 0$ and $\frac{n}{n+p} < q < 2$, $T$ is bounded from $\dot{F}^{0,q}_{1,1}$ to $\dot{F}^{0,q}_{b,1}$ if and only $T b \in \dot{F}^{0,q}_{b,\infty}$. Also, note that [22, Theorem 1.1] is a special case of Theorem 1.7 for $b \equiv 1$.

Our result is based on the discrete Calderón-type reproducing formula [10, Theorem 2.11], the characterization of Triebel–Lizorkin spaces $\dot{F}^{0,q}_{b,p}$ [15, Theorem 4.2], and a Plancherel–Pôlya-type inequality [15, Theorem 3.1]. We first define the paraproduct operator, similar to what the second named author did in [21,22], and prove the boundedness of paraproduct operators acting from $\dot{F}^{0,q}_{1,p}$ to $\dot{F}^{0,q}_{b,p}$. We then apply an argument similar to the one in [2] to obtain our $Tb$ theorem which extends previous results in [3,9,10,12,15,22].

This paper is organized as follows. In Section 2, we give some preliminaries. Then we study paraproduct operators in Section 3. Finally we prove the main theorem in Section 4. Through the paper, we use $Q$ to denote a dyadic cube in $\mathbb{R}^n$ and use $C$ to denote a positive constant independent of the main variables, which may vary from line to line.
2. Preliminaries

For the convenience of readers, we summarize some known results established in [9,10,15], which we will need in the sequel.

Recall the definition and properties of homogeneous Triebel–Lizorkin spaces of para-accretive type, and start with “test functions” given by Han [9]. Fix two exponents $0 < \beta \leq 1$ and $\gamma > 0$. Suppose that $b$ is a para-accretive function. A function $f$ defined on $\mathbb{R}^n$ is said to be a test function of type $(\beta, \gamma, b)$ centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$ if

\[
\left| f(x) \right| \leq C \frac{d^\beta}{(d + |x - x_0|)^{n+\gamma}}; \\
\left| f(x) - f(x') \right| \leq C \left( \frac{|x - x'|}{d + |x - x_0|} \right)^\beta \frac{d^\gamma}{(d + |x - x_0|)^{n+\gamma}} \quad \text{for} \quad |x - x'| \leq \frac{d + |x - x_0|}{2}; \\
\int_{\mathbb{R}^n} f(x) b(x) \, dx = 0.
\]

Denote by $\mathcal{M}^{(\beta, \gamma, b)}(x_0, d)$ the collection of all test functions of type $(\beta, \gamma, b)$ centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$. For $f \in \mathcal{M}^{(\beta, \gamma, b)}(x_0, d)$, the norm of $f$ in $\mathcal{M}^{(\beta, \gamma, b)}(x_0, d)$ is defined by

\[
\|f\|_{\mathcal{M}^{(\beta, \gamma, b)}(x_0, d)} := \inf \{ C : (4) \text{ and } (5) \text{ hold} \}.
\]

We denote $\mathcal{M}^{(\beta, \gamma, b)}(0, 1)$ simply by $\mathcal{M}^{(\beta, \gamma, b)}$. Clear that $\mathcal{M}^{(\beta, \gamma, b)}$ is a Banach space under the norm $\|f\|_{\mathcal{M}^{(\beta, \gamma, b)}}$. Write

\[
b \mathcal{M}^{(\beta, \gamma, b)} := \{ f : f = bg \text{ for some } g \in \mathcal{M}^{(\beta, \gamma, b)} \}.
\]

If $f \in b \mathcal{M}^{(\beta, \gamma, b)}$ and $f = bg$ for $g \in \mathcal{M}^{(\beta, \gamma, b)}$, then the norm of $f$ is defined by $\|f\|_{b \mathcal{M}^{(\beta, \gamma, b)}} := \|g\|_{\mathcal{M}^{(\beta, \gamma, b)}}$. As usual, we use $(\mathcal{M}^{(\beta, \gamma, b)})'$ and $(b \mathcal{M}^{(\beta, \gamma, b)})'$ to denote the dual spaces of $\mathcal{M}^{(\beta, \gamma, b)}$ and $b \mathcal{M}^{(\beta, \gamma, b)}$, respectively. Use $\langle h, f \rangle$ to denote the natural pairing of elements $f \in (\mathcal{M}^{(\beta, \gamma, b)})'$ and $f \in \mathcal{M}^{(\beta, \gamma, b)}$. It is easy to check that for any $x_0 \in \mathbb{R}^n$ and $d > 0$, $\mathcal{M}^{(\beta, \gamma, b)}(x_0, d) = \mathcal{M}^{(\beta, \gamma, b)}(0, 1)$ with equivalent norms. Thus, given $h \in (\mathcal{M}^{(\beta, \gamma, b)})'$, $\langle h, f \rangle$ is well defined for all $f \in \mathcal{M}^{(\beta, \gamma, b)}(x_0, d)$ with any $x_0 \in \mathbb{R}^n$ and $d > 0$.

In order to state the Calderón reproducing formula, we also need an approximation to the identity (cf. [3,9]).

**Definition 2.1.** Let $b$ be a para-accretive function. A sequence of operators $\{S_k\}_{k \in \mathbb{Z}}$ is called an approximation to the identity associated to $b$ if the kernels $S_k(x, y)$ of $S_k$ are functions from $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{C}$ such that there exist constant $C$ and some $0 < \varepsilon \leq 1$ satisfying, for all $k \in \mathbb{Z}$ and all $x, x', y, y' \in \mathbb{R}^n$,

(i) \[ |S_k(x, y)| \leq C \frac{2^{-ke}}{(2^{-k} + |x - y|)^{n+\varepsilon}}, \]

(ii) \[ |S_k(x, y) - S_k(x', y)| \leq C \left( \frac{|x - x'|}{2^{-k} + |x - y|} \right)^{\varepsilon} \frac{2^{-ke}}{(2^{-k} + |x - y|)^{n+\varepsilon}}, \quad \text{for} \quad |x - x'| \leq \frac{1}{2} (2^{-k} + |x - y|), \]

(iii) \[ |S_k(x, y) - S_k(x, y')| \leq C \left( \frac{|y - y'|}{2^{-k} + |x - y|} \right)^{\varepsilon} \frac{2^{-ke}}{(2^{-k} + |x - y|)^{n+\varepsilon}}, \quad \text{for} \quad |y - y'| \leq \frac{1}{2} (2^{-k} + |x - y|), \]

(iv) \[ |\left[ S_k(x, y) - S_k(x, y') \right] - [S_k(x', y) - S_k(x', y')] | \leq C \left( \frac{|x - x'|}{2^{-k} + |x - y|} \right)^{\varepsilon} \left( \frac{|y - y'|}{2^{-k} + |x - y|} \right)^{\varepsilon} \frac{2^{-ke}}{(2^{-k} + |x - y|)^{n+\varepsilon}}, \]

\[ \text{for} \quad |x - x'| \leq \frac{1}{2} (2^{-k} + |x - y|) \quad \text{and} \quad |y - y'| \leq \frac{1}{2} (2^{-k} + |x - y|). \]

(v) \[ \int_{\mathbb{R}^n} S_k(x, y) b(y) \, dy = 1 \quad \text{for all} \quad k \in \mathbb{Z} \quad \text{and} \quad x \in \mathbb{R}^n, \]

(vi) \[ \int_{\mathbb{R}^n} S_k(x, y) b(x) \, dx = 1 \quad \text{for all} \quad k \in \mathbb{Z} \quad \text{and} \quad y \in \mathbb{R}^n. \]

The following discrete Calderón reproducing formulae were given in [10].
Proposition 2.2. Suppose that \( \{S_k\} \) is an approximation to the identity defined in Definition 2.1. Set \( D_k = S_k - S_{k-1} \). Then there exists a family of operators \( \{D^*_k\} \) with kernel \( \mathcal{D}^*_k(x, y) \) satisfying, for \( 0 < \varepsilon < \varepsilon' < \varepsilon \),

\[
|D^*_k(x, y)| \leq C \frac{2^{-k\varepsilon'}}{(2^k + |x - y|)^{n+\varepsilon'}},
\]

\[
|D^*_k(x, y) - D^*_k(x, y')| \leq C \left( \frac{|y - y'|}{2^k + |x - y|} \right)^{\varepsilon'} \frac{2^{-k\varepsilon'}}{(2^k + |x - y|)^{n+\varepsilon'}} \quad \text{for } |y - y'| \leq (2^{-k} + |x - y|)/2,
\]

\[
\int_{\mathbb{R}^n} D^*_k(x, y) b(y) \, dy = 0 \quad \text{for all } k \in \mathbb{Z} \text{ and } x \in \mathbb{R}^n,
\]

\[
\int_{\mathbb{R}^n} D^*_k(x, y) b(x) \, dx = 0 \quad \text{for all } k \in \mathbb{Z} \text{ and } y \in \mathbb{R}^n,
\]

such that,

\[
f(x) = \sum_k \sum_Q D_k f(y_Q) \int_Q b(x) D^*_k(y, x) b(y) \, dy \quad \text{for all } f \in (\mathcal{M}^{(\beta, \gamma, b)}),
\]

and

\[
f(x) = \sum_k \sum_Q D_k b f(y_Q) \int_Q D^*_k(y, x) b(y) \, dy \quad \text{for all } f \in (b\mathcal{M}^{(\beta, \gamma, b)}),
\]

where \( Q \) are all dyadic cubes with the side length \( 2^{-k-N} \) for some fixed positive large integer \( N \) and \( y_Q \) is any fixed point in \( Q \).

Note that \( f \in (b\mathcal{M}^{(\beta, \gamma, b)})' \) if and only if \( b f \in (\mathcal{M}^{(\beta, \gamma, b)})' \) or equivalently, \( f \in (\mathcal{M}^{(\beta, \gamma, b)})' \) if and only if \( b^{-1} f \in (b\mathcal{M}^{(\beta, \gamma, b)})' \).

The classical Plancherel–Pólya inequality has a long history and plays a central role in the theory of function spaces. Roughly speaking, if a tempered distribution \( f \) in \( \mathbb{R}^n \), whose Fourier transform has compact support, then, by the Paley-Wiener theorem, it is an analytic function, or more precisely, entire analytic function of exponential type. The Plancherel-Pólya inequality concludes that if \( \{x_k\} \) is an appropriate set of points in \( \mathbb{R}^n \), e.g. lattice points, where the length of the mesh is sufficiently small, then

\[
\left( \sum_{k=1}^{\infty} |f(x_k)|^p \right)^{1/p} \approx \|f\|_p
\]

for all \( 0 < p < \infty \) with a modification if \( p = \infty \). The Fourier transform is the basic tool to prove such an inequality. See [20] for more details.

For any cube \( Q \) and \( \lambda > 0 \), we denote by \( \lambda Q \) the cube concentric with \( Q \) whose each edge is \( \lambda \) times as long. A generalized Plancherel–Pólya-type inequality was given in [15, Theorem 3.1].

Proposition 2.3. Let \( \alpha \in (-\varepsilon, \varepsilon) \), \( \max\left\{ \frac{n}{n+\varepsilon}, \frac{n}{n+\varepsilon+\alpha} \right\} < p < \infty \) and \( \max\left\{ \frac{n}{n+\varepsilon}, \frac{n}{n+\varepsilon+\alpha} \right\} < q \leq \infty \). Suppose that \( \{S_k\}_{k \in \mathbb{Z}} \) and \( \{P_k\}_{k \in \mathbb{Z}} \) are approximations to the identity defined in Definition 2.1. Set \( D_k = S_k - S_{k-1} \) and \( E_k = P_k - P_{k-1} \). Then, for any real number \( \lambda \geq 1 \),

(i) for all \( f \in (b\mathcal{M}^{(\beta, \gamma, b)})' \),

\[
\left\| \sum_k \sum_{Q_k} \left( 2^{k\alpha} \sup_{z \in \lambda Q_k} |E_k b f(z)| \right)^q \chi_{Q_k} \right\|_p \approx \left\| \sum_k \sum_{Q_k} \left( 2^{k\alpha} \inf_{z \in \lambda Q_k} |D_k b f(z)| \right)^q \chi_{Q_k} \right\|_p ;
\]

(ii) for all \( f \in (\mathcal{M}^{(\beta, \gamma, b)})' \),

\[
\left\| \sum_k \sum_{Q_k} \left( 2^{k\alpha} \sup_{z \in \lambda Q_k} |E_k f(z)| \right)^q \chi_{Q_k} \right\|_p \approx \left\| \sum_k \sum_{Q_k} \left( 2^{k\alpha} \inf_{z \in \lambda Q_k} |D_k f(z)| \right)^q \chi_{Q_k} \right\|_p .
\]

In the case \( p = \infty \),
(iii) for all \( f \in (b, M^{(\beta, \gamma, b)})' \),
\[
\sup_P \left\{ |P|^{-1} \int P \left( \sum_{k=-\log_2 \ell(P)}^\infty \sum_{Q_k \subseteq P} \left( 2^{k\alpha} \sup_{z \in Q_k} |D_k b f (z)| \right)^q \chi_{Q_k} (x) \right) \right\}^{1/q} = \sup_P \left\{ |P|^{-1} \int P \left( \sum_{k=-\log_2 \ell(P)}^\infty \sum_{Q_k \subseteq P} \left( 2^{k\alpha} \inf_{z \in Q_k} |D_k b f (z)| \right)^q \chi_{Q_k} (x) \right) \right\}^{1/q};
\]

(iv) for all \( f \in (M^{(\beta, \gamma, b)})' \),
\[
\sup_P \left\{ |P|^{-1} \int P \left( \sum_{k=-\log_2 \ell(P)}^\infty \sum_{Q_k \subseteq P} \left( 2^{k\alpha} \sup_{z \in Q_k} |E_k b f (z)| \right)^q \chi_{Q_k} (x) \right) \right\}^{1/q} = \sup_P \left\{ |P|^{-1} \int P \left( \sum_{k=-\log_2 \ell(P)}^\infty \sum_{Q_k \subseteq P} \left( 2^{k\alpha} \inf_{z \in Q_k} |D_k b f (z)| \right)^q \chi_{Q_k} (x) \right) \right\}^{1/q}.
\]

Note that we can regard the \( \varepsilon \)'s in both Definitions 1.1 and 2.1 to be the same by choosing the smaller one. Suppose that \( b \) is a para-accretive function and \( \{S_k\}_{k \in \mathbb{Z}} \) is an approximation to the identity defined in Definition 2.1. Set \( D_k = S_k - S_{k-1} \). For \( \alpha \in (-\varepsilon, \varepsilon) \), \( 0 < q \leq +\infty \), and \( f \in (b, M^{(\beta, \gamma, b)})' \), the \( g \)-function and \( S \)-function associated to a para-accretive function \( b \) is defined by
\[
S_b^{\alpha, q} (f) (x) = \left\{ \sum_k 2^{k\alpha} |D_k b f (x)|^q \right\}^{1/q},
\]
\[
S_b^{\alpha, q} (f) (x) = \left\{ \sum_k \int_{|x-y| \leq 2^{-k}} 2^{k\alpha} |D_k b f (y)|^q \, dy \right\}^{1/q}.
\]

When \( q = \infty \), the above two definitions should be modified to the \( \ell^\infty \)-norms as usual.

Similar to the classical case, we have the equivalent \( L^p \)-norms for \( g \)-function and \( S \)-function as follows (cf. [15, Theorem 3.4]). For \( \alpha \in (-\varepsilon, \varepsilon) \), \( \max\{ \frac{n}{\beta + \gamma}, \frac{n}{\beta + \gamma + \varepsilon} \} < p < \infty \) and \( \max\{ \frac{n}{\beta + \gamma}, \frac{n}{\beta + \gamma + \varepsilon} \} < q \leq \infty \),
\[
\left\| S_b^{\alpha, q} (f) \right\|_p \approx \left\| g_b^{\alpha, q} (f) \right\|_p.
\]

Thus, a class of the homogeneous Triebel–Lizorkin spaces associated to para-accretive functions can be defined as follows.

**Definition 2.4.** Suppose that \( b \) is a para-accretive function, \( \{S_k\}_{k \in \mathbb{Z}} \) is an approximation to the identity defined in Definition 2.1, and \( D_k = S_k - S_{k-1} \). For \( \alpha \in (-\varepsilon, \varepsilon) \), \( \max\{ \frac{n}{\beta + \gamma}, \frac{n}{\beta + \gamma + \varepsilon} \} < p < \infty \) and \( \max\{ \frac{n}{\beta + \gamma}, \frac{n}{\beta + \gamma + \varepsilon} \} < q \leq \infty \), the homogeneous Triebel–Lizorkin space of para-accretive type \( L^{\alpha, q}_{\beta, p} \) is the collection of \( f \in (b, M^{(\beta, \gamma, b)})' \) such that
\[
\left\| f \right\|_{L^{\alpha, q}_{\beta, p}} := \left\| S_b^{\alpha, q} (f) \right\|_p < \infty.
\]

When \( p = \infty \), we define the space by
\[
\left\| f \right\|_{L^{\alpha, q}_{\beta, \infty}} := \sup_P \left\{ \frac{1}{|P|} \int P \left( \sum_{k=-\log_2 \ell(P)}^\infty 2^{k\alpha} |D_k b f (x)|^q \right) \right\}^{1/q} < \infty,
\]
where the supremum is taken over all dyadic cubes \( P \) in \( \mathbb{R}^n \).

By Proposition 2.3, the norm of a distribution in a homogeneous Triebel–Lizorkin space of para-accretive type is independent of the choice of the approximation to the identity. By [9, (3.7) and Theorem 3.10] and [4, Definitions 2.1 and 3.1], we also have \( F_{\beta, p}^{\alpha, q} = b^{-1} F_{\beta, p}^{\alpha, q} \) and \( L^{\alpha, q}_{\beta, 1} = b^{-1} L^{\alpha, q}_{\beta, 1} \). Indeed, if \( f \in b F_{\beta, p}^{\alpha, q} \) means \( \| f \|_{b F_{\beta, p}^{\alpha, q}} = \| (\sum (2^{k\alpha} |D_k f |)q/4)^{1/q} \|_p \) which implies that \( \| f \|_{b F_{\beta, p}^{\alpha, q}} = \| g_{\alpha} (f) \|_p = \| f \|_{L^{\alpha, q}_{\beta, 1}} \). Alternatively, if \( f \in b^{-1} F_{\beta, p}^{\alpha, q} \) means \( b f \in F_{\beta, 1}^{\alpha, q} \) which implies that \( \| f \|_{b^{-1} F_{\beta, p}^{\alpha, q}} = \| b f \|_{F_{\beta, p}^{\alpha, q}} = \| g_{\alpha} (f) \|_p = \| f \|_{L^{\alpha, q}_{\beta, 1}} \). Note that, for \( \frac{n}{\beta + \gamma} < p \leq 1 \), \( \| f \|_{L^{\alpha, q}_{\beta, p}} = H_{b}^{p} \).
Let us recall corresponding sequence spaces introduced by Frazier and Jawerth [6, p. 38 and p. 40]. For $0 < p, q < +\infty$, $\alpha \in \mathbb{R}$, and a sequence $s = \{s_k\}$, define
\[
\|s\|_{j_p^{\alpha,q}} := \left\{ \sum_{k} \sum_{Q_k} \left| Q_k \right| \left| s_k \right|^{1/2-\alpha/n} |s_{Q_k}| (x) \right\}^{1/q}, \quad \text{if } p < +\infty,
\]
and
\[
\|s\|_{j_\infty^{\alpha,q}} := \sup_{p} \left\{ \frac{1}{|Q_k|} \int_{Q_k} \sum_{k \in \mathbb{R}^n} \left| Q_k \right| \left| s_k \right|^{1/2-\alpha/n} |s_{Q_k}| (x) \right\}^{1/q},
\]
where the sums and the supremum involved $P$ or $Q_k$, respectively, are taken over all dyadic cubes in $\mathbb{R}^n$. As a consequence of Proposition 2.3 and (10), we have the following characterization.

**Proposition 2.5.** (See [15] and 2.1) Suppose that $b$ is a para-accretive function, $\{s_k\}_{k \in \mathbb{Z}}$ is an approximation to the identity defined in Definition 2.1, and $D_k = S_k - S_k-1$. Let $\alpha \in (-\varepsilon, \varepsilon)$, $\max\left\{ \frac{n-\varepsilon}{n+\varepsilon+\alpha}, \frac{n+\varepsilon+\alpha}{n-\varepsilon+\alpha} \right\} < p, q < +\infty$, and $y_{Q_k}$ be any fixed point in $Q_k$.

(i) For $p < \infty$,
\[
\|f\|_{F_p^{\alpha,q}} \approx \|s^{\alpha,q}_b(f)\|_{L_p} \approx \|Q_k\|^{1/2} D_k bf(y_{Q_k})\|_{L_p^{\alpha,q}},
\]
\[
\|f\|_{F_1^{\alpha,q}} \approx \|s^{\alpha,q}_1(f)\|_{L_1} \approx \|Q_k\|^{1/2} D_k f(y_{Q_k})\|_{L_1^{\alpha,q}}.
\]
(ii) For $p = \infty$,
\[
\|f\|_{F_\infty^{\alpha,q}} \approx \|Q_k\|^{1/2} D_k bf(y_{Q_k})\|_{L_\infty^{\alpha,q}},
\]
\[
\|f\|_{F_1^{\alpha,q}} \approx \|Q_k\|^{1/2} D_k f(y_{Q_k})\|_{L_1^{\alpha,q}}.
\]
(iii) If $f \in F_1^{\alpha,q}$, then $b^{-1} f \in F_1^{\alpha,q}$ and $\|f\|_{F_1^{\alpha,q}} \approx \|b^{-1} f\|_{F_1^{\alpha,q}}$.
(iv) If $f \in F_1^{\alpha,q}$, then $bf \in F_1^{\alpha,q}$ and $\|bf\|_{F_1^{\alpha,q}} \approx \|bf\|_{F_1^{\alpha,q}}$.
(v) If $1 < p, q < +\infty$, then the dual space of $F_1^{\alpha,q}$ is $\tilde{F}_1^{1,q'}$ and the dual space of $F_1^{\alpha,q}$ is $\tilde{F}_1^{\alpha,1}$.

### 3. Paraproduct operators

Before studying the boundedness of singular integral operators, we recall the definition of paraproduct operators associated to a para-accretive function and find the assertion for the boundedness of such operators acting on Triebel–Lizorkin spaces of para-accretive type. Choose a non-negative function $\Phi \in \mathcal{D}(\mathbb{R}^n)$ satisfying $\int \Phi(x) \, dx = 1$ and $\text{supp}(\Phi) \subseteq [0, 1]^n$. For each cube $Q$, let $\Phi_Q = |Q|^{-1/2} \Phi((x - y_{Q_k})/|Q|)$, where $y_{Q_k}$ is a fixed point in $Q$.

For a para-accretive function $b, h \in \tilde{F}_{1,\infty}^{n,q}$ and $f \in (\mathcal{M}^{(p,y)})'$, the $b$-paraproduct of $h$ and $f$ is defined by
\[
\Pi_h^{(b)}(f)(x) = \sum_{k} \sum_{Q_k} \left| Q_k \right|^{-1/2} D_k bh(y_{Q_k}) \left| f, b^{-1} \Phi_Q \right| \int_{Q_k} D_k^z(z, x)b(z) \, dz
\]
\[
= \int \Pi_h^{(b)}(x, y) b^{-1}(y) f(y) \, dy,
\]
where $D_k$ and $D_k^z$ are given in Proposition 2.2, $y_{Q_k}$ is a fixed point in $Q_k$, and its kernel is
\[
K_h^{(b)}(x, y) = \sum_{k} \sum_{Q_k} \left| Q_k \right|^{-1/2} D_k bh(y_{Q_k}) \Phi_Q (y) \int_{Q_k} D_k^z(z, x)b(z) \, dz.
\]

**Remark 3.1.** Let $h \in \tilde{F}_{1,\infty}^{n,q}$ and define the paraproduct operator $\Pi_h$ by
\[
\Pi_h(f) = \sum_{Q} |Q|^{-1/2} \left( \psi_Q, h \right) (f, \Phi_Q) \psi_Q,
\]
where \( f = \sum_q (f, \varphi_q)\psi_q \) is the \( \varphi \)-transform identity given by Frazier and Jawerth in [6]. It was shown in [22] that \( \Pi_h \in \text{SIO}(1), \Pi_h \in \text{WBP}, \Pi_h1 = h \) and \( \Pi_h1 = 0. \) By a similar argument, it can be proved that \( \Pi_h^{(b)} \in \text{GSI}(\varepsilon), M_b\Pi_h^{(b)} M_b \in \text{WBP}, \Pi_h^{(b)} b = h \) and \( \Pi_h^{(b)} b = 0. \)

It follows from (11) and Proposition 2.2 that, for \( f \in \hat{L}^{0,q}_{q,1,p} \),

\[
\Pi_h^{(b)}(f)(x) = \sum_k \sum_{Q_k} |Q_k|^{-1/2} D_k bh(y_{Q_k}) \int_{Q_k} D_k(y, x) b(y) dy
\]

\[
= \sum_k \sum_{Q_k} |Q_k|^{-1/2} D_k bh(y_{Q_k}) \sum_j D_j f(y_{Q_j}) \left( \int_{Q_j} D_j(y, \cdot) b(y) dy, \Phi_{Q_j} \right) \int_{Q_j} D_j(y, x) b(y) dy.
\]

Let \( s = \{ s_{Q_j} \} = |Q_j|^{-1/2} D_j f(y_{Q_j}) \), \( d = \{ d_{Q_k} \} = |Q_k|^{-1/2} D_k bh(y_{Q_k}) \), and \( T_d \) be defined by \((T_d s)_{Q_k} = |Q_k|^{-1/2} d_{Q_k} s_{Q_k}.\) Also let \( G^{(b)} \) denote the matrix corresponding to

\[
G_{Q_k, Q_j}^{(b)} = \left[ \left| Q_j \right|^{-1/2} \int_{Q_j} D_j(y, \cdot) b(y) dy, \Phi_{Q_j} \right].
\]

By Proposition 2.5,

\[
\left\| \Pi_h^{(b)}(f) \right\|_{L^{0,q}_{b,1,p}} \approx \left\| T_d G^{(b)} s \right\|_{L^{0,q}_{b,1,p}}.
\]

Therefore, to prove the boundedness of \( \Pi_h^{(b)} \), we need to check the boundedness of matrices \( T_d \) and \( G^{(b)}.\) The matrix \( T_d \) is
diagonal but the matrix \( G^{(b)} \) fails the almost diagonality criterion (cf. [7] for classical case).

More preliminaries are needed. Let us recall the definition of almost diagonality given in [6]. For \( \alpha \in \mathbb{R} \) and \( 0 < p, q \leq \infty, \) we say that a matrix \( A = \{ a_{Q_k, P_j} \} \) is \((\alpha, p, q)\)-almost diagonal if there exists \( \varepsilon > 0 \) such that

\[
\sup_{Q_k, P_j} \frac{|a_{Q_k, P_j}|}{w_{Q_k, P_j}(\varepsilon)} < +\infty,
\]

where

\[
w_{Q_k, P_j}(\varepsilon) = \left( \frac{\ell(Q_k)}{\ell(P_j)} \right)^{\alpha} \left( 1 + \frac{|y_{Q_k} - y_{P_j}|}{\max\{\ell(P_j), \ell(Q_k)\}} \right)^{-j-\varepsilon} \min\left\{ \left( \frac{\ell(Q_k)}{\ell(P_j)} \right)^{n/2+\varepsilon/2}, \left( \frac{\ell(P_j)}{\ell(Q_k)} \right)^{\varepsilon/2+j-n/2} \right\}
\]

for \( J = n/\min\{1, p, q\} \), fixed points \( y_{Q_k} \) and \( y_{P_j} \) in \( Q_k \) and \( P_j \), respectively.

**Lemma 3.2.** Let \( G^{(b)} \) be an operator with matrix defined in (12).

(a) If \( \frac{n}{p+\alpha} < p < \infty, \) then \( G^{(b)} : \tilde{f}^{0,2} \rightarrow \tilde{f}^{0,\infty} \) is bounded.

(b) If \(-\varepsilon < \alpha < 0, 0 < p < \infty, \) and \( 0 < q \leq \infty, \) then \( G^{(b)} \) is \((\alpha, p, q)\)-almost diagonal and hence bounded on \( \tilde{f}^{0,q}.\)

**Proof.** Part (a) was proved in [15]. For (b), the proof is similar to the proof of [21, Lemma 2.4] by using [6, Lemma B.1] and conditions (6)--(9). \( \square \)

We also need the following embedding theorem.

**Theorem 3.3 (Embedding theorem).**

(a) For \( \alpha \in (-\varepsilon, \varepsilon), \) \( \max\{\frac{n}{p+\varepsilon}, \frac{n}{\frac{n}{p+\varepsilon}+\frac{n}{p+\alpha}}\} < p \leq +\infty, \) and \( \max\{\frac{n}{p+\varepsilon}, \frac{n}{\frac{n}{p+\varepsilon}+\frac{n}{p+\alpha}}\} \leq q_1 \leq q_2 \leq +\infty, \) the following embeddings are continuous

\[
\tilde{f}^{\alpha, q_1}_{b, p} \hookrightarrow \tilde{f}^{\alpha, q_2}_{b, p} \quad \text{and} \quad \tilde{f}^{\alpha, q_1}_{p} \hookrightarrow \tilde{f}^{\alpha, q_2}_{p}.
\]

(b) For \( \alpha_1, \alpha_2 \in (-\varepsilon, \varepsilon), \alpha_1 > \alpha_2, \) \( \max\{\frac{n}{p+\varepsilon}, \frac{n}{\frac{n}{p+\varepsilon}+\frac{n}{p+\alpha}}\} < p_1 < p_2 < +\infty, \) \( \alpha_1 - \frac{n}{p_1} = \alpha_2 - \frac{n}{p_2} \) and \( \max\{\frac{n}{p+\varepsilon}, \frac{n}{\frac{n}{p+\varepsilon}+\frac{n}{p+\alpha}}\} \leq q_1, q_2 \leq +\infty, \) the following embeddings are continuous

\[
\tilde{f}^{\alpha_1, q_1}_{b, p_1} \hookrightarrow \tilde{f}^{\alpha_2, q_2}_{b, p_2} \quad \text{and} \quad \tilde{f}^{\alpha_1, q_1}_{p_1} \hookrightarrow \tilde{f}^{\alpha_2, q_2}_{p_2}.
\]
Proof. Part (a) follows from the embedding $\ell^{q_1} \hookrightarrow \ell^{q_2}$. The proof of (b) is followed immediately by Propositions 2.2 and 2.5. □

Theorem 3.4 (The boundedness of paraproduct $\Pi^{(b)}_h$).

(a) If $\frac{n}{n+\varepsilon} < q \leq 2$, $\frac{n}{n+\varepsilon} < p < \infty$, and $h \in \ell^{q,\infty}$, then $\Pi^{(b)}_h \in \text{GSIO}(\varepsilon)$, $M_b \Pi^{(b)}_h M_b \in \text{WBP}$, $\Pi^{(b)}_h b = h$, $(\Pi^{(b)}_h)^* b = 0$, and $\Pi^{(b)}_h$ extends to a bounded operator from $\hat{F}^{0,q}_{1,p}$ to $\hat{F}^{0,q}_{b,p}$ with

$$\|\Pi^{(b)}_h(f)\|_{\hat{F}^{0,q}_{b,p}} \leq C \|h\|_{\ell^{q,\infty}} \|f\|_{\hat{F}^{0,q}_{1,p}} \quad \text{for all } f \in \hat{F}^{0,q}_{1,p}.$$

(b) If $2 < q \leq +\infty$, $\frac{n}{n+\varepsilon} < p < \infty$, and $h \in U_b := \bigcup_{0<\varepsilon<\ell} \hat{F}^{0,\infty}_{b,n/\delta}$, then $\Pi^{(b)}_h \in \text{GSIO}(\varepsilon)$, $M_b \Pi^{(b)}_h M_b \in \text{WBP}$, $\Pi^{(b)}_h b = h$, $(\Pi^{(b)}_h)^* b = 0$, and $\Pi^{(b)}_h$ extends to a bounded operator from $\hat{F}^{0,q}_{1,p}$ to $\hat{F}^{0,q}_{b,p}$ with

$$\|\Pi^{(b)}_h(f)\|_{\hat{F}^{0,q}_{b,p}} \leq C \|h\|_{\ell^{q,\infty}} \|f\|_{\hat{F}^{0,q}_{1,p}} \quad \text{for all } f \in \hat{F}^{0,q}_{1,p} \text{ and all } 0<\delta<\varepsilon.$$

Proof. Part (a) was proved in [15, Theorem 5.4]. For (b), from definitions it is easy to obtain $\hat{F}^{0,\infty}_{b,n/\delta} \hookrightarrow \hat{F}^{0,\infty}_{b,\infty}$. Hence $\Pi^{(b)}_h \in \text{GSIO}(\varepsilon)$, $M_b \Pi^{(b)}_h M_b \in \text{WBP}$, $\Pi^{(b)}_h b = h$ and $(\Pi^{(b)}_h)^* b = 0$ by Remark 3.1. So we only have to show that $\Pi^{(b)}_h$ is bounded from $\hat{F}^{0,q}_{1,p}$ to $\hat{F}^{0,q}_{b,p}$.

By Theorem 3.3 (b), Lemma 3.2 (b) and [21, Lemma 2.3], we have, for any $0<\beta<1$,

$$\hat{F}^{0,q}_{p,\beta} \hookrightarrow \hat{F}^{n(1-\beta)/p,q/\beta}_{p/\beta} \overset{(\Pi^{(b)}_h)}{\hookrightarrow} \hat{F}^{n(1-\beta)/p,q/\beta}_{p/\beta} \overset{T_p}{\hookrightarrow} \hat{F}^{0,q}_{p,\beta}$$

provided $d \in \bigcup_{\beta>0} \hat{F}^{n(1-\beta)/p,q/\beta}_{p/\beta}$. But by Proposition 2.5 and Theorem 3.3,

$$\|d\|_{\hat{F}^{n(1-\beta)/p,q/\beta}_{p/\beta}} \approx \|h\|_{\hat{F}^{n(1-\beta)/p,q/\beta}_{p/\beta}} \leq C \|h\|_{\ell^{q,\infty}_{b,n/\delta}}$$

if $\delta > n(1-\beta)/p$. So

$$\|\Pi^{(b)}_h(f)\|_{\hat{F}^{0,q}_{b,p}} \leq C \|d\|_{\hat{F}^{n(1-\beta)/p,q/\beta}_{p/\beta}} \cdot \|s\|_{\hat{F}^{0,q}_{b,p}} \leq C \|h\|_{\ell^{q,\infty}_{b,n/\delta}} \cdot \|f\|_{\hat{F}^{0,q}_{1,p}}.$$

Thus, we finish the proof. □

By Proposition 2.5 and Theorem 3.4, if $\Pi^{(b)}_h$ is bounded from $\hat{F}^{0,q}_{1,p}$ to $\hat{F}^{0,q}_{b,p}$, then $M_b \Pi^{(b)}_h$ is bounded from $\hat{F}^{0,q}_{1,p}$ to $\hat{F}^{0,q}_{b,p}$ and $\Pi^{(b)}_h M_b$ is bounded from $\hat{F}^{0,q}_{b,p}$ to $\hat{F}^{0,q}_{b,p}$.

Theorem 3.5.

(a) For $2 \leq q \leq \infty$ and $1 < p < \infty$, if $g \in \hat{F}^{0,q}_{b,\infty}$, then $(\Pi^{(b)}_g)^* \overset{\text{duality}}{=} \text{extends to a bounded operator from } \hat{F}^{0,q}_{1,p} \text{ to } \hat{F}^{0,q}_{b,p}$ with

$$\|\Pi^{(b)}_g(f)\|_{\hat{F}^{0,q}_{b,p}} \leq C \|g\|_{\ell^{q,\infty}} \|f\|_{\hat{F}^{0,q}_{1,p}} \quad \text{for all } f \in \hat{F}^{0,q}_{1,p}.$$

(b) For $\beta > 0$, $0 < p < \infty$, and $0 < q \leq \infty$, if $g \in \hat{F}^{0,\infty}_{b,n/\beta}$, then $(\Pi^{(b)}_g)^* \overset{\text{duality}}{=} \text{extends to a bounded operator from } \hat{F}^{0,q}_{1,p} \text{ to } \hat{F}^{0,q}_{b,p}$ with

$$\|\Pi^{(b)}_g(f)\|_{\hat{F}^{0,q}_{b,p}} \leq C \|g\|_{\ell^{q,\infty}} \|f\|_{\hat{F}^{0,q}_{1,p}} \quad \text{for all } f \in \hat{F}^{0,q}_{1,p}.$$

Proof. By Theorem 3.4 and duality, (a) follows immediately. For (b), by (11) the adjoint operator of $\Pi^{(b)}_g$ is

$$(\Pi^{(b)}_g)^* f(x) = \sum_{f \in P_j} \int \Phi_{P_j}(x) \left( \int D_j y ight) (y, x) dy ight) \Phi_{P_j}(x)$$

and $\|\Pi^{(b)}_g\|_{\hat{F}^{0,q}_{b,p}} \approx \|\Pi^{(b)}_g\|_{\hat{F}^{0,q}_{b,p}} \approx \|\Pi^{(b)}_g\|_{\hat{F}^{0,q}_{b,p}}$, where $D_{k\ell}Q_k(x) = \int D_{k\ell}^2(y, x) b(y) dy$ by Proposition 2.5. Let $d_{\ell,j} = D_{k\ell} b(y) dy$, $d = [d_{\ell,j}]$, and $A = [\{\Phi_{P_j}, D_{k\ell}Q_k\}]$. Then, similar to (13), $\|\Pi^{(b)}_g\|_{\hat{F}^{0,q}_{b,p}} \approx \|ATd\|_{\hat{F}^{0,q}_{b,p}}$, where $s = \{(f, D_{k\ell}, p)\}$. Let $D = ([D_{\ell,R}, D_{\ell,j}, 1, t_{\ell,R}], \ell_{\ell,R}, D_{\ell,R} b(y) dy)$ and $t = (t_{\ell,R})$. Then $D$ is almost diagonal and $s = D t$. Hence $\|s\|_{\hat{F}^{0,q}_{b,p}} \approx \|ATd\|_{\hat{F}^{0,q}_{b,p}} \approx \|\Pi^{(b)}_g\|_{\hat{F}^{0,q}_{b,p}}$. 

||f||_{L^p_{b,\beta}}. Since \(A\) is the transpose of \(G^{(b)}\), by Lemma 3.2, \(A\) is \((\beta, p, q)\) almost diagonal and hence \(A\) is bounded on \(\dot{F}_{p}^{\beta,q}\). By [21, Lemma 2.3] and Theorem 3.3, the composition

\[
\dot{F}_{p}^{\beta,q} \xrightarrow{\mathcal{T}_{g_{\beta}}} \dot{F}_{p_1}^{\beta,q} \xrightarrow{A} \dot{F}_{p_1}^{\beta,q} \xrightarrow{\mathcal{T}_{g_{\beta}}} \dot{F}_{p}^{\beta,q}
\]

is bounded if \(d \in \dot{F}_{n/\beta}^{\beta,\infty}\), where \(p_1\) satisfies \(n/p = -\beta + n/p_1\). By Proposition 2.5, \(d \in \dot{F}_{n/\beta}^{\beta,\infty}\) is equivalent to \(g \in \dot{F}_{b,n/\beta}^{\beta,\infty}\).

We already have the sufficient conditions for the boundedness of \(\Pi_{h}^{(b)}\). A necessary condition for the boundedness of \(\Pi_{h}^{(b)}\) was provided by the authors in [15].

**Proposition 3.6.** Let \(\frac{n}{p+\varepsilon} < p < +\infty\) and \(\frac{n}{p+\varepsilon} < q \leq 2\). If \(\Pi_{h}^{(b)}\) is bounded from \(\dot{F}_{1,p}^{\beta,q}\) to \(\dot{F}_{b,p}^{\beta,q}\), then \(h \in \dot{F}_{b,\infty}^{\beta,q}\).

4. **Proof of the main theorem**

**Proof of Theorem 1.7.** Let \(h = T_{b}\), \(g = T_{b}^{*}b\) and \(S = T - \Pi_{h}^{(b)} - (\Pi_{S}^{(b)})^{*}\). By Theorems 3.4 and 3.5, \(S\) satisfies \(S \in GSO(e)\), \(S \in WB\), \(S_{b} = 0\) and \(S^{*}b = 0\). Thus, by [15, Theorem 5.1], \(S\) is bounded from \(\dot{F}_{1,p}^{\beta,q}\) to \(\dot{F}_{b,p}^{\beta,q}\). It remains to prove that \(\Pi_{h}^{(b)}\) and \((\Pi_{S}^{(b)})^{*}\) are bounded on \(\dot{F}_{1,\beta}^{0,q}\) to \(\dot{F}_{b,\beta}^{0,q}\).

(a) For \(q = 2\), by Theorems 3.4(a) and 3.5(a), \(\Pi_{h}^{(b)}\) and \((\Pi_{S}^{(b)})^{*}\) are bounded from \(\dot{F}_{1,2}^{0,q}\) to \(\dot{F}_{b,2}^{0,q}\), since \(h, g \in \dot{F}_{b,\infty}^{0,q}\).

(b) If \(T_{b} = h \in \dot{F}_{b,\infty}^{0,q}\) for \(1 \leq q < 2\), then \(\Pi_{h}^{(b)}\) is bounded from \(\dot{F}_{1,\beta}^{0,q}\) to \(\dot{F}_{b,\beta}^{0,q}\) by Theorem 3.4(a). If \(T_{b}^{*}b = g \in U_{b}\), then \((\Pi_{S}^{(b)})^{*}\) is bounded from \(\dot{F}_{1,\beta}^{0,q}\) to \(\dot{F}_{b,\beta}^{0,q}\) due to Theorem 3.5(b).

(c) This follows from (b) by duality.

The arguments for cases (d)–(h) are similar, and we leave details to readers. \(\square\)

**References**


