# ON THE HODGE-TYPE DECOMPOSITION AND COHOMOLGY GROUPS OF $k$-CAUCHY-FUETER COMPLEXES OVER DOMAINS IN THE QUATERNIONIC SPACE

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ABSTRACT. The $k$-Cauchy-Fueter operator $D_0^{(k)}$ on one dimensional quaternionic space $\mathbb{H}$ is the Euclidean versions of helicity $\frac{k}{2}$ massless field operator on the Minkowski space. The $k$-Cauchy-Fueter equation for $k \geq 2$ is overdetermined and the compatibility condition is given by the $k$-Cauchy-Fueter complex. These complexes play the role of Dolbeault complex in several complex variables. We prove that a natural boundary value problem associated to this complex is regular. Then by using the theory for regular boundary value problems, we show the Hodge-type orthogonal decomposition, and the fact that the non-homogeneous $k$-Cauchy-Fueter equation $D_0^{(k)}u = f$ on a smooth domain $\Omega$ in $\mathbb{H}$ is solvable if and only if $f$ satisfies the compatibility condition and is orthogonal to the set $\mathscr{H}^1_{(k)}(\Omega)$ of Hodge-type elements. This set is isomorphic to the first cohomology group of the $k$-Cauchy-Fueter complex over $\Omega$, which is finite dimensional, while the second cohomology group is always trivial.

## 1. Introduction

On one dimensional quaternionic space, the $k$-Cauchy-Fueter operators are the Euclidean version of helicity $\frac{k}{2}$ massless field operators [10] [22] on the Minkowski space (corresponding to the Dirac-Weyl equation for $k = 1$, Maxwell's equation for $k = 2$, the linearized Einstein's equation for $k = 3$, etc.). They are the quaternionic counterpart of the Cauchy-Riemann operator in complex analysis. The $k$-Cauchy-Fueter complexes on multidimensional quaternionic space $\mathbb{H}^n$, which play the role of Dolbeault complex in several complex variables, are now explicitly known [20] (see also [2] [3] [6] [7]). It is quite interesting to develop a theory of several quaternionic variables by analyzing these complexes, as analyzing Dolbeault complex in the theory of several complex variables. A well known theorem in several complex variables states that the Dolbeault cohomology of a domain vanishes if and only if it is pseudoconvex. Many remarkable results about holomorphic functions can be deduced by considering non-homogeneous $\overline{\partial}$-equations (cf., e.g., [11]), which leads to the study of $\overline{\partial}$-Neumann problem. We have solved [20] the non-homogeneous $k$-Cauchy-Fueter equation on the whole quaternionic space $\mathbb{H}^n$ and deduced Hartogs' phenomenon and integral representation formulae. See [13] [14] [17] [20] (also [1] [4] [5] [7] [21] for $k = 1$) and references therein for results about $k$-regular functions.

Note that the non-homogeneous $\overline{\partial}$-equation on a smooth domain in the complex plane is always solvable. In our case the non-homogeneous 1-Cauchy-Fueter equation on a smooth domain in $\mathbb{H}$ is always solvable since it is exactly the Dirac operator on $\mathbb{R}^4$. But even on one dimensional quaternionic space $\mathbb{H}$, the $k$-Cauchy-Fueter operator for $k \geq 2$ is overdetermined. The non-homogeneous $k$-Cauchy-Fueter equation only can be solved under the compatibility condition given by the $k$-Cauchy-Fueter complex. The $k$-Cauchy-Fueter complex over a smooth domain $\Omega$

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in $\mathbb{H}$ is

$$0 \longrightarrow C^\infty(\Omega,\mathbb{C}^{k+1}) \xrightarrow{D_0^{(k)}} C^\infty(\Omega,\mathbb{C}^{2k}) \xrightarrow{D_1^{(k)}} C^\infty(\Omega,\mathbb{C}^{k-1}) \longrightarrow 0, \tag{1.1}$$

$k = 2, 3, \ldots$, where $D_0^{(k)}$ is the $k$-Cauchy-Fueter operator. In this paper, we will investigate the non-homogeneous $k$-Cauchy-Fueter equation

$$D_0^{(k)} u = f, \tag{1.2}$$

on a smooth domain $\Omega$ in $\mathbb{H}$ under the compatibility condition

$$D_1^{(k)} f = 0. \tag{1.3}$$

We define the first cohomology group of the $k$-Cauchy-Fueter complex as

$$H^1_{(k)}(\Omega) = \frac{\left\{f \in C^\infty(\Omega;\mathbb{C}^{2k}); D_1^{(k)} f = 0\right\}}{\left\{D_0^{(k)} u; u \in C^\infty(\Omega;\mathbb{C}^{k+1})\right\}},$$

and the second cohomology group as

$$H^2_{(k)}(\Omega) = \frac{C^\infty(\Omega;\mathbb{C}^{k-1})}{\left\{D_1^{(k)} u; u \in C^\infty(\Omega;\mathbb{C}^{2k})\right\}}.$$

The 0-th cohomology group as $H^0_{(k)}(\Omega) = \ker D_0^{(k)}$. This is the space of $k$-regular functions, the dimension of which is infinite (cf. [13]).

The first cohomology group can be represented by Hodge-type elements:

$$\mathscr{H}^1_{(k)}(\Omega) = \left\{f \in C^\infty(\Omega,\mathbb{C}^{2k}); D_1^{(k)} f = 0, D_0^{(k)*} f = 0\right\},$$

where $D_0^{(k)*}$ is the formal adjoint of $D_0^{(k)}$.

Let $H^s(\Omega)$ be the usual Sobolev space of complex valued functions, defined on a domain $\Omega$. Denote by $H^s(\Omega,\mathbb{C}^n)$ the space of all $\mathbb{C}^n$-valued functions, whose components are in $H^s(\Omega)$.

**Theorem 1.1.** *Suppose $\Omega$ is a domain in $\mathbb{H}$ with smooth boundary. Then*
*(1) The isomorphic spaces*

$$H^1_{(k)}(\Omega) \cong \mathscr{H}^1_{(k)}(\Omega)$$

*are finite dimensional.*
*(2) If $f \in H^s(\Omega,\mathbb{C}^{2k})$ $(s = 1, 2, \ldots)$, then the non-homogeneous $k$-Cauchy-Fueter equation (1.2) is solvable by some $u \in H^{s+1}(\Omega,\mathbb{C}^{k+1})$ if and only if $f$ is orthogonal to $\mathscr{H}^1_{(k)}(\Omega)$ in $L^2(\Omega,\mathbb{C}^{2k})$ and satisfies the compatibility condition (1.3). When it is solvable, it has a solution $u$ satisfying the estimate*

$$\|u\|_{H^{s+1}(\Omega,\mathbb{C}^{k+1})} \le C\|f\|_{H^s(\Omega,\mathbb{C}^{2k})}, \tag{1.4}$$

*for some constant $C$ only depending on the domain $\Omega$, $k$ and $s$.*
*(3) The equation*

$$D_1^{(k)} \psi = \Psi,$$

*is uniquely solved by a $\psi \in H^{s+1}(\Omega,\mathbb{C}^{2k})$ for any $\Psi \in H^s(\Omega,\mathbb{C}^{k-1})$ with estimate as (1.4).*

By (3), the second cohomology group always vanishes. To prove Theorem 1.1, we consider the associated Laplacian of the complex (1.1)

$$\Box_1^{(k)} = D_0^{(k)} D_0^{(k)*} + D_1^{(k)*} D_1^{(k)}, \tag{1.5}$$

where $D_0^{(k)*}$ and $D_1^{(k)*}$ be the formal adjoints of $D_0^{(k)}$ and $D_1^{(k)}$, respectively, and a natural boundary value problem

$$\begin{cases} \Box_1^{(k)} u = f, & \text{on} \quad \Omega, \\ D_0^{(k)*}(\nu)u\Big|_{\partial\Omega} = 0, & \\ D_1^{(k)*}(\nu)D_1^{(k)}u\Big|_{\partial\Omega} = 0, & \end{cases} \tag{1.6}$$

where $\nu$ is the unit vector of outer normal to the boundary, $u \in H^{s+2}(\Omega, \mathbb{C}^{2k})$ and $f \in H^s(\Omega, \mathbb{C}^{2k})$. We prove that this boundary value problem is regular and obtain the following result.

**Theorem 1.2.** *Suppose $\Omega$ is a domain in $\mathbb{H}$ with a smooth boundary. If $f \in H^s(\Omega, \mathbb{C}^{2k})$ $(s = 0, 1, 2, \ldots)$ is orthogonal to $\mathscr{H}^1_{(k)}(\Omega)$ under the $L^2$ inner product, the boundary value problem (1.6) has a solution $u = N_1^{(k)} f$ such that*

$$\|u\|_{H^{s+2}(\Omega,\mathbb{C}^{2k})} \leq C\|f\|_{H^s(\Omega,\mathbb{C}^{2k})} \tag{1.7}$$

*for some constant $C$ only depending on the domain $\Omega$, $k$ and $s$.*

*Moreover, we have the Hodge-type orthogonal decomposition for any $\psi \in H^s(\Omega, \mathbb{C}^{2k})$:*

$$\psi = D_0^{(k)} D_0^{(k)*} N_1^{(k)} \psi + D_1^{(k)*} D_1^{(k)} N_1^{(k)} \psi + P\psi, \tag{1.8}$$

*where $P$ is the orthonomal projection to $\mathscr{H}^1_{(k)}(\Omega)$ under the $L^2(\Omega, \mathbb{C}^{2k})$ inner product.*

Although for a smooth domain in the complex plane, its Dolbeault cohomology always vanishes, its De Rham cohomolgy groups, which are isomorphic to its simplicial cohomolog groups, may be nontrivial. We conjecture that the cohomology groups $H^1_{(k)}(\Omega)$ may be nontrivial for some domains $\Omega$ with smooth boundaries in $\mathbb{H}$. It is quite interesting to characterize the class of domains in $\mathbb{H}$ on which the non-homogeneous $k$-Cauchy-Fueter equation is always solvable. On the higher dimensional quaternionic space $\mathbb{H}^n$, there is no reason to expect the corresponding boundary value problem of the non-homogeneous $k$-Cauchy-Fueter equation to be regular, as in the case of several complex variables. The problem becomes much harder. It is also interesting to find some $L^2$ estimates for the $k$-Cauchy-Fueter equation on a domain in $\mathbb{H}^n$.

In section 2, we will write the 2-Cauchy-Fueter operator $D_0^{(2)}$ and the operator $D_1^{(2)}$ explicitly as a $(4 \times 3)$-matrix and a $(1 \times 4)$-matrix valued differential operators of first order with constant coefficients, respectively, and calculate the associated Laplacian. We also find the natural boundary conditions for functions in domains of the adjoint operator $D_0^{(2)*}$ or $D_1^{(2)*}$. In section 3, we prove that the boundary value problem (1.6) satisfies the Shapiro-Lopatinskii condition, i.e., it is a regular boundary value problem. In section 4, we generalize the results of sections 2 and 3 to the cases $k \geq 3$. The $k$-Cauchy-Fueter operator $D_0^{(k)}$ and the second operator $D_1^{(k)}$ in the complex (1.1) are written explicitly as matrix valued differential operators of first order with constant coefficients, the associated Laplacians are calculated, and the boundary value problem is proved to be also regular. In section 5, we apply the general theory for elliptic boundary value problems to show that $\Box_1^{(k)}$ is a Fredholm operator between suitable Sobolev spaces. This implies the Hodge-type decomposition and allow us to prove main theorems.

Because we only work on one dimensional quaternionic space, the result in [20] about the $k$-Cauchy-Fueter complexes we will use later can be proved by elementary method. So this paper is self-contained.

## 2. The $k$-Cauchy-Fueter operators

2.1. **The $k$-Cauchy-Fueter complexes on a domain in $\mathbb{H}$.** We will identify the one dimensional quaternionic space $\mathbb{H}$ with the Euclidean space $\mathbb{R}^4$. Set

$$\left(\begin{array}{cc} \nabla_{00'} & \nabla_{01'} \\ \nabla_{10'} & \nabla_{11'} \end{array}\right) := \left(\begin{array}{cc} \partial_{x_0}+i\partial_{x_1} & -\partial_{x_2}-i\partial_{x_3} \\ \partial_{x_2}-i\partial_{x_3} & \partial_{x_0}-i\partial_{x_1} \end{array}\right), \tag{2.1}$$

where $(x_0,x_1,x_2,x_3)\in\mathbb{R}^4$. The matrix

$$\epsilon=(\epsilon_{A'B'})=\left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right) \tag{2.2}$$

is used to raise or lower indices, e.g. $\nabla_A^{A'}\epsilon_{A'B'}=\nabla_{AB'}$.

The *$k$-Cauchy-Fueter complex* [20] on a domain $\Omega$ in $\mathbb{R}^4$ for $k\geq 2$ is

$$0\longrightarrow C^\infty(\Omega,\odot^k\mathbb{C}^2)\xrightarrow{D_0^{(k)}}C^\infty(\Omega,\odot^{k-1}\mathbb{C}^2\otimes\mathbb{C}^2)\xrightarrow{D_1^{(k)}}C^\infty(\Omega,\odot^{k-2}\mathbb{C}^2\otimes\Lambda^2\mathbb{C}^2)\longrightarrow 0, \tag{2.3}$$

where

$$\begin{aligned} (D_0^{(k)}\phi)_{AB'\cdots C'} &:= \sum_{A'=0',1'}\nabla_A^{A'}\phi_{A'B'\cdots C'}, \\ (D_1^{(k)}\psi)_{ABB'\cdots C'} &:= \sum_{A'=0',1'}\left(\nabla_A^{A'}\psi_{BA'B'\cdots C'}-\nabla_B^{A'}\psi_{AA'B'\cdots C'}\right). \end{aligned} \tag{2.4}$$

Here a section $\phi\in C^\infty(\Omega,\odot^k\mathbb{C}^2)$ has $(k+1)$ components $\phi_{0'\ldots0'},\phi_{1'\ldots0'},\ldots,\phi_{1'\ldots1'}$, while $D_0^{(k)}\phi\in C^\infty(\Omega,\odot^{k-1}\mathbb{C}^2\otimes\mathbb{C}^2)$ has $2k$ components $(D_0^{(k)}\phi)_{A0'\ldots0'}$, $(D_0^{(k)}\phi)_{A1'\ldots0'},\ldots,(D_0^{(k)}\phi)_{A1'\ldots1'}$, where $A=0,1$. Note that $\phi_{A'B'\ldots C'}$ is invariant under the permutation of subscripts, $A',B',\cdots,C'=0',1'$.

In physics, there are a family of equations called the *helicity $\frac{k}{2}$ massless field* equations [10] [22]. The first one is the Dirac-Weyl equation of an electron for mass zero whose solutions correspond to neutrinos. The second one is the Maxwell's equation whose solutions correspond to photons. The third one is the linearized Einstein's equation whose solutions correspond to weak gravitational field, and so on. The $k$-Cauchy-Fueter equations are the Euclidean version of these equations. The *affine Minkowski space* can be embedded in $\mathbb{C}^{2\times2}$ by

$$(x_0,x_1,x_2,x_3)\mapsto\left(\begin{array}{cc} x_0+x_1 & x_2+ix_3 \\ x_2-ix_3 & x_0-x_1 \end{array}\right), \tag{2.5}$$

$i=\sqrt{-1}$, while the quaternionic algebra $\mathbb{H}$ can be embedded in $\mathbb{C}^{2\times2}$ by

$$x_0+x_1\mathbf{i}+x_2\mathbf{j}+x_3\mathbf{k}\mapsto\left(\begin{array}{cc} x_0+ix_1 & -x_2-ix_3 \\ x_2-ix_3 & x_0-ix_1 \end{array}\right). \tag{2.6}$$

The helicity $\frac{k}{2}$ massless field equations (cf. [10] [20]) is

$$D_0^{(k)}\phi=0,$$

where the $D_0^{(k)}$ is also given by (2.4) with $\nabla_{AB'}$ replaced by

$$\left(\begin{array}{cc} \nabla_{00'} & \nabla_{01'} \\ \nabla_{10'} & \nabla_{11'} \end{array}\right) := \left(\begin{array}{cc} \partial_{x_0}+\partial_{x_1} & \partial_{x_2}+i\partial_{x_3} \\ \partial_{x_2}-i\partial_{x_3} & \partial_{x_0}-\partial_{x_1} \end{array}\right). \tag{2.7}$$

2.2. **The 2-Cauchy-Fueter complex.** We write

$$\begin{aligned}\begin{pmatrix}\nabla_0^{0'} & \nabla_0^{1'}\\ \nabla_1^{0'} & \nabla_1^{1'}\end{pmatrix} &= \begin{pmatrix}\nabla_{00'} & \nabla_{01'}\\ \nabla_{10'} & \nabla_{11'}\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}\nabla_{01'} & -\nabla_{00'}\\ \nabla_{11'} & -\nabla_{10'}\end{pmatrix}\\ &= \begin{pmatrix}-\partial_{x_2} - i\partial_{x_3} & -\partial_{x_0} - i\partial_{x_1}\\ \partial_{x_0} - i\partial_{x_1} & -\partial_{x_2} + i\partial_{x_3}\end{pmatrix}.\end{aligned} \tag{2.8}$$

In the case $k = 2$, we use the notation $D_0 = D_0^{(2)}$ and $D_1 = D_1^{(2)}$. The 2-Cauchy-Fueter complex on a domain $\Omega$ in $\mathbb{R}^4$ is

$$0 \longrightarrow C^\infty(\Omega, \odot^2\mathbb{C}^2) \xrightarrow{D_0} C^\infty(\Omega, \mathbb{C}^2 \otimes \mathbb{C}^2) \xrightarrow{D_1} C^\infty(\Omega, \Lambda^2\mathbb{C}^2) \longrightarrow 0, \tag{2.9}$$

where

$$\begin{aligned}(D_0\phi)_{AB'} &:= \sum_{A'=0',1'} \nabla_A^{A'}\phi_{A'B'} = \nabla_A^{0'}\phi_{0'B'} + \nabla_A^{1'}\phi_{1'B'},\\ (D_1\psi)_{01} &:= \sum_{A'=0',1'} \nabla_0^{A'}\psi_{1A'} - \nabla_1^{A'}\psi_{0A'} = \nabla_0^{0'}\psi_{10'} + \nabla_0^{1'}\psi_{11'} - \nabla_1^{0'}\psi_{00'} - \nabla_1^{1'}\psi_{01'},\end{aligned} \tag{2.10}$$

where $A = 0, 1, B' = 0', 1'$, $\phi \in C^\infty(\Omega, \odot^2\mathbb{C}^2)$ has 3 components $\phi_{0'0'}, \phi_{1'0'} = \phi_{0'1'}$ and $\phi_{1'1'}$, while $D_0\phi \in C^\infty(\Omega, \mathbb{C}^2 \otimes \mathbb{C}^2)$ has 4 components $(D_0\phi)_{00'}$,$(D_0\phi)_{10'}$, $(D_0\phi)_{01'}$ and $(D_0\phi)_{11'}$, and $\Psi = \Psi_{01} \in C^\infty(\Omega, \Lambda^2\mathbb{C}^2)$ is a scalar function.

We know from results in [20] that (2.9) is a complex: $D_1D_0 = 0$. It is can be checked directly as follows. For any $\phi \in C^\infty(\Omega, \odot^2\mathbb{C}^2)$, we calculate

$$\begin{aligned}(D_1D_0\phi)_{01} &= \sum_{A'=0',1'} \nabla_0^{A'}(D_0\phi)_{1A'} - \nabla_1^{A'}(D_0\phi)_{0A'}\\ &= \sum_{A',C'=0',1'} \nabla_0^{A'}\nabla_1^{C'}\phi_{C'A'} - \nabla_1^{A'}\nabla_0^{C'}\phi_{C'A'} = 0\end{aligned} \tag{2.11}$$

by $\phi_{C'A'} = \phi_{A'C'}$ and the commutativity $\nabla_1^{A'}\nabla_0^{C'} = \nabla_0^{C'}\nabla_1^{A'}$, as scalar differential operators of constant coefficients.

The operator $D_0$ in (2.9) can be written as a $(4 \times 3)$-matrix operator

$$D_0\phi = \begin{pmatrix}(D_0\phi)_{00'}\\ (D_0\phi)_{10'}\\ (D_0\phi)_{01'}\\ (D_0\phi)_{11'}\end{pmatrix} = \begin{pmatrix}\nabla_0^{0'} & \nabla_0^{1'} & 0\\ \nabla_1^{0'} & \nabla_1^{1'} & 0\\ 0 & \nabla_0^{0'} & \nabla_0^{1'}\\ 0 & \nabla_1^{0'} & \nabla_1^{1'}\end{pmatrix}\begin{pmatrix}\phi_{0'0'}\\ \phi_{1'0'}\\ \phi_{1'1'}\end{pmatrix},$$

and the operator $D_1$ takes the form

$$D_1\psi = (-\nabla_1^{0'}, \nabla_0^{0'}, -\nabla_1^{1'}, \nabla_0^{1'})\begin{pmatrix}\psi_{00'}\\ \psi_{10'}\\ \psi_{01'}\\ \psi_{11'}\end{pmatrix}.$$

Define

$$z_0 = x_0 + ix_1, \qquad z_1 = x_2 + ix_3$$

and

$$\begin{aligned}\partial_{z_0} &= \partial_{x_0} - i\partial_{x_1}, &\qquad \partial_{\overline{z}_0} &= \partial_{x_0} + i\partial_{x_1},\\ \partial_{z_1} &= \partial_{x_2} - i\partial_{x_3}, &\qquad \partial_{\overline{z}_1} &= \partial_{x_2} + i\partial_{x_3}.\end{aligned}$$

Our notations coincide with the usual ones up to a factor $\frac{1}{2}$. Using these notations, and the following isomorphisms

$$\odot^2\mathbb{C}^2 \cong \mathbb{C}^3, \qquad \mathbb{C}^2\otimes\mathbb{C}^2 \cong \mathbb{C}^4, \qquad \Lambda^2\mathbb{C}^2 \cong \mathbb{C}^1,$$

we can rewrite $D_0 : C^\infty(\Omega,\mathbb{C}^3) \to C^\infty(\Omega,\mathbb{C}^4)$ with

$$D_0\phi = \begin{pmatrix} -\partial_{\overline{z}_1} & -\partial_{\overline{z}_0} & 0 \\ \partial_{z_0} & -\partial_{z_1} & 0 \\ 0 & -\partial_{\overline{z}_1} & -\partial_{\overline{z}_0} \\ 0 & \partial_{z_0} & -\partial_{z_1} \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \end{pmatrix},$$

and $D_1 : C^\infty(\Omega,\mathbb{C}^4) \to C^\infty(\Omega,\mathbb{C})$ with

$$D_1\psi = (-\partial_{z_0}, -\partial_{\overline{z}_1}, \partial_{z_1}, -\partial_{\overline{z}_0}) \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}.$$

2.3. **The associated Laplacian.** It is easy to see that

(2.12)
$$\overline{\begin{pmatrix} -\partial_{\overline{z}_1} & -\partial_{\overline{z}_0} \\ \partial_{z_0} & -\partial_{z_1} \end{pmatrix}}^t \begin{pmatrix} -\partial_{\overline{z}_1} & -\partial_{\overline{z}_0} \\ \partial_{z_0} & -\partial_{z_1} \end{pmatrix} = \begin{pmatrix} -\partial_{z_1} & \partial_{\overline{z}_0} \\ -\partial_{z_0} & -\partial_{\overline{z}_1} \end{pmatrix} \begin{pmatrix} -\partial_{\overline{z}_1} & -\partial_{\overline{z}_0} \\ \partial_{z_0} & -\partial_{z_1} \end{pmatrix} = \begin{pmatrix} \triangle & 0 \\ 0 & \triangle \end{pmatrix},$$

where $^t$ is the transpose, and

$$\Delta := \partial_{z_0}\partial_{\overline{z}_0} + \partial_{z_1}\partial_{\overline{z}_1} = \partial^2_{x_0} + \partial^2_{x_1} + \partial^2_{x_2} + \partial^2_{x_4}$$

is the usual Laplacian on $\mathbb{R}^4$.

Let $\mathscr{D} : C^1(\overline{\Omega},\mathbb{C}^{n_1}) \to C^0(\overline{\Omega},\mathbb{C}^{n_2})$ be a differential operator of the first order with constant coefficients. An operator $\mathscr{D}^*$ is called the *formal adjoint* of $\mathscr{D}$ if for any $u \in C^1_0(\Omega,\mathbb{C}^{n_1})$, $v \in C^1_0(\Omega,\mathbb{C}^{n_2})$, we have

$$\int_\Omega \langle \mathscr{D}u, v\rangle dV = \int_\Omega \langle u, \mathscr{D}^* v\rangle dV,$$

where $\langle\cdot,\cdot\rangle$ is the Hermitian inner product in $\mathbb{C}^{n_j}$, $j = 1, 2$. It is easy to see that the formal adjoints of $D_0$ and $D_1$ are $D_0^* = -\overline{D_0}^t$ and $D_1^* = -\overline{D_1}^t$, respectively. Then,

(2.13)
$$\begin{aligned} D_0D_0^* &= -\begin{pmatrix} -\partial_{\overline{z}_1} & -\partial_{\overline{z}_0} & 0 \\ \partial_{z_0} & -\partial_{z_1} & 0 \\ 0 & -\partial_{\overline{z}_1} & -\partial_{\overline{z}_0} \\ 0 & \partial_{z_0} & -\partial_{z_1} \end{pmatrix} \begin{pmatrix} -\partial_{z_1} & \partial_{\overline{z}_0} & 0 & 0 \\ -\partial_{z_0} & -\partial_{\overline{z}_1} & -\partial_{z_1} & \partial_{\overline{z}_0} \\ 0 & 0 & -\partial_{z_0} & -\partial_{\overline{z}_1} \end{pmatrix} \\ &= -\begin{pmatrix} \Delta & 0 & \partial_{\overline{z}_0}\partial_{z_1} & -\partial^2_{\overline{z}_0} \\ * & \Delta & \partial^2_{z_1} & -\partial_{\overline{z}_0}\partial_{z_1} \\ * & * & \Delta & 0 \\ * & * & * & \Delta \end{pmatrix}, \end{aligned}$$

where $*$-entries are known by Hermitian symmetry of $D_0^*D_0$, and

$$\begin{aligned} D_1^*D_1 &= -\begin{pmatrix} -\partial_{\overline{z}_0} \\ -\partial_{z_1} \\ \partial_{\overline{z}_1} \\ -\partial_{z_0} \end{pmatrix} (-\partial_{z_0}, -\partial_{\overline{z}_1}, \partial_{z_1}, -\partial_{\overline{z}_0}) \\ &= -\begin{pmatrix} \partial_{z_0}\partial_{\overline{z}_0} & \partial_{\overline{z}_0}\partial_{\overline{z}_1} & -\partial_{\overline{z}_0}\partial_{z_1} & \partial_{\overline{z}_0}^2 \\ * & \partial_{z_1}\partial_{\overline{z}_1} & -\partial_{z_1}^2 & \partial_{\overline{z}_0}\partial_{z_1} \\ * & * & \partial_{z_1}\partial_{\overline{z}_1} & -\partial_{\overline{z}_0}\partial_{\overline{z}_1} \\ * & * & * & \partial_{z_0}\partial_{\overline{z}_0} \end{pmatrix}. \end{aligned} \tag{2.14}$$

The sum of (2.13) and (2.14) gives

$$\begin{aligned} \Box_1 := D_0D_0^* + D_1^*D_1 &= -\begin{pmatrix} \Delta + \partial_{z_0}\partial_{\overline{z}_0} & \partial_{\overline{z}_0}\partial_{\overline{z}_1} & 0 & 0 \\ \partial_{z_0}\partial_{z_1} & \Delta + \partial_{z_1}\partial_{\overline{z}_1} & 0 & 0 \\ 0 & 0 & \Delta + \partial_{z_1}\partial_{\overline{z}_1} & -\partial_{\overline{z}_0}\partial_{\overline{z}_1} \\ 0 & 0 & -\partial_{z_0}\partial_{z_1} & \Delta + \partial_{z_0}\partial_{\overline{z}_0} \end{pmatrix} \\ &= -\begin{pmatrix} \Delta + \Delta_1 & L & 0 & 0 \\ \overline{L} & \Delta + \Delta_2 & 0 & 0 \\ 0 & 0 & \Delta + \Delta_2 & -L \\ 0 & 0 & -\overline{L} & \Delta + \Delta_1 \end{pmatrix} \end{aligned} \tag{2.15}$$

which is obviously elliptic (i.e., its symbol for any $\xi \neq 0$ is positive definite), where

$$\begin{aligned} \Delta_1 &:= \partial_{z_0}\partial_{\overline{z}_0} = \partial_{x_0}^2 + \partial_{x_1}^2, \\ \Delta_2 &:= \partial_{z_1}\partial_{\overline{z}_1} = \partial_{x_2}^2 + \partial_{x_3}^2, \\ L &:= \partial_{\overline{z}_0}\partial_{\overline{z}_1} = (\partial_{x_0} + i\partial_{x_1})(\partial_{x_2} + i\partial_{x_3}). \end{aligned}$$

2.4. **Domains of the adjoint operators $D_0^*$ and $D_1^*$.** We define the inner product on $L^2(\Omega, \mathbb{C}^n)$ by

$$(u, v) = \int_\Omega \langle u, v\rangle dV,$$

where $\langle \cdot, \cdot \rangle$ is the Hermitian inner product in $\mathbb{C}^n$, $dV$ is the Lebesgue measure.

For a differential operator $\mathscr{D} : C^1(\overline{\Omega}, \mathbb{C}^{n_1}) \to C^0(\overline{\Omega}, \mathbb{C}^{n_2})$ of the first order with constant coefficients, $u \in C^1(\overline{\Omega}, \mathbb{C}^{n_1})$ and $v \in C^1(\overline{\Omega}, \mathbb{C}^{n_2})$, we have

$$\int_\Omega \langle \mathscr{D}u, v\rangle dV = \int_\Omega \langle u, \mathscr{D}^* v\rangle dV + \int_{\partial\Omega} \langle u, \mathscr{D}^*(\nu) v\rangle dS, \tag{2.16}$$

by Green's formula, where $\nu = (\nu_0, \ldots, \nu_4)$ is the unit vector of outer normal to the boundary, and $\mathscr{D}^*(\nu)$ is obtained by replacing $\partial_{x_j}$ in $\mathscr{D}^*$ by $\nu_j$.

By abuse of notations, we denote also by $\mathscr{D}^*$ the adjoint operator of $\mathscr{D} : L^2(\Omega, \mathbb{C}^{n_1}) \to L^2(\Omega, \mathbb{C}^{n_2})$. Now let $\Omega$ be $\mathbb{R}^4_+ = \{x = (x_0, \ldots, x_3) \in \mathbb{R}^4; x_0 > 0\}$. Then the unit inner normal vector is $\nu = (1, 0, 0, 0)$. By definition of the adjoint operator, a function $\psi = (\psi_0, \psi_1, \psi_2, \psi_3)^t$ $\in \mathrm{Dom}D_0^* \cap C^1(\Omega, \mathbb{C}^4)$ if and only if the integral over the boundary in (2.16) vanishes for any $u$, i.e., $D_0^*(\nu)\psi = 0$ on the boundary. Then,

$$0 = \begin{pmatrix} -\partial_{z_1} & \partial_{\overline{z}_0} & 0 & 0 \\ -\partial_{z_0} & -\partial_{\overline{z}_1} & -\partial_{z_1} & \partial_{\overline{z}_0} \\ 0 & 0 & -\partial_{z_0} & -\partial_{\overline{z}_1} \end{pmatrix} (\nu)\psi|_{\partial\Omega} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \psi|_{\partial\Omega},$$

from which we get

$$\psi_1 = \psi_2 = 0, \qquad \psi_0 - \psi_3 = 0 \qquad \text{on} \quad \partial\Omega. \tag{2.17}$$

Similarly, $\Psi \in \mathrm{Dom}D_1^* \cap C^1(\Omega, \mathbb{C})$ if and only if $D_1^*(\nu)\Psi = 0$ on the boundary, i.e.,

$$0 = \begin{pmatrix} -\partial_{\bar{z}_0} \\ -\partial_{z_1} \\ \partial_{\bar{z}_1} \\ -\partial_{z_0} \end{pmatrix} (\nu)\Psi|_{\partial\Omega} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \Psi|_{\partial\Omega},$$

from which we get $\Psi|_{\partial\Omega} = 0$. Now $D_1\psi \in \mathrm{Dom}D_1^* \cap C^1(\Omega, \mathbb{C})$ implies that

$$-\partial_{z_0}\psi_0 - \partial_{\bar{z}_1}\psi_1 + \partial_{z_1}\psi_2 - \partial_{\bar{z}_0}\psi_3 = 0, \qquad \text{on} \quad \partial\Omega.$$

Note that $\partial_{\bar{z}_1}\psi_1 = \partial_{z_1}\psi_2 = 0$ since $\partial_{\bar{z}_1}$ and $\partial_{z_1}$ are tangential derivatives, and $\psi_1$, $\psi_2$ both vanish on the boundary by using (2.17). Therefore,

$$\partial_{z_0}\psi_0 + \partial_{\bar{z}_0}\psi_3 = \partial_{x_0}(\psi_0 + \psi_3) = 0, \qquad \text{on} \quad \partial\Omega \tag{2.18}$$

by using (2.17) again. So we need to solve the system $\Box_1^{(2)}\psi = f$ in $\Omega$ under the boundary conditions (2.17) and (2.18).

We need to define more operators. We obtain $\Box_0 := D_0^* D_0$ equals to

$$-\begin{pmatrix} -\partial_{z_1} & \partial_{\bar{z}_0} & 0 & 0 \\ -\partial_{z_0} & -\partial_{\bar{z}_1} & -\partial_{z_1} & \partial_{\bar{z}_0} \\ 0 & 0 & -\partial_{z_0} & -\partial_{\bar{z}_1} \end{pmatrix} \begin{pmatrix} -\partial_{\bar{z}_1} & -\partial_{\bar{z}_0} & 0 \\ \partial_{z_0} & -\partial_{z_1} & 0 \\ 0 & -\partial_{\bar{z}_1} & -\partial_{\bar{z}_0} \\ 0 & \partial_{z_0} & -\partial_{z_1} \end{pmatrix} = -\begin{pmatrix} \Delta & 0 & 0 \\ 0 & 2\Delta & 0 \\ 0 & 0 & \Delta \end{pmatrix}, \tag{2.19}$$

and

$$\Box_2 := D_1 D_1^* = -(-\partial_{z_0}, -\partial_{\bar{z}_1}, \partial_{z_1}, -\partial_{\bar{z}_0}) \begin{pmatrix} -\partial_{\bar{z}_0} \\ -\partial_{z_1} \\ \partial_{\bar{z}_1} \\ -\partial_{z_0} \end{pmatrix} = -2\Delta, \tag{2.20}$$

with the boundary condition $\Psi \in \mathrm{Dom}D_1^* \cap C^1(\Omega, \mathbb{C}^1)$, i.e., the Dirichlet condition $\Psi|_{\partial\Omega} = 0$.

**Corollary 2.1.** *Suppose that $u \in H^1(\Omega, \mathbb{C}^{n_1})$, $v \in H^1(\Omega, \mathbb{C}^{n_2})$, and $\mathscr{D}(\nu)u|_{\partial\Omega} = 0$ or $\mathscr{D}^*(\nu)v|_{\partial\Omega} = 0$. Then,*

$$(\mathscr{D}u, v) = (u, \mathscr{D}^*v), \qquad (v, \mathscr{D}u) = (\mathscr{D}^*v, u) \tag{2.21}$$

*Proof.* The trace theorem states that the operator of restriction to the boundary $H^s(\Omega, \mathbb{C}^n) \to H^{s-\frac{1}{2}}(\partial\Omega, \mathbb{C}^n)$ for $s > \frac{1}{2}$ is a bounded operator (cf. Proposition 4.5 in chapter 4 in [16]). Moreover, $C^\infty(\overline{\Omega}, \mathbb{C}^n)$ is dense in $H^s(\Omega, \mathbb{C}^n)$ for $s \geq 0$. Approximating $u \in H^1(\Omega, \mathbb{C}^{n_1})$, $v \in H^1(\Omega, \mathbb{C}^{n_2})$, by functions from $C^\infty(\overline{\Omega}, \mathbb{C}^{n_j})$, we see that integration by part (2.16) holds for $u \in H^1(\Omega, \mathbb{C}^{n_1})$, $v \in H^1(\Omega, \mathbb{C}^{n_2})$, (cf. (7.2) in chapter 5 in [16]). The boundary term vanishes by the assumption. □

## 3. The Shapiro-Lopatinskii condition

3.1. **The Shapiro-Lopatinskii condition.** Suppose that $P(x, \partial) : C^\infty(\overline{\Omega}, E_0) \to C^\infty(\overline{\Omega}, E_1)$ is an elliptic differential operator of order $m$, and that $B_j(x, \partial) : C^\infty(\overline{\Omega}, E_0) \to C^\infty(\partial\Omega, G_j)$, $j = 1, \ldots, l$, are differential operators of order $m_j \leq m - 1$, where $E_0, E_1, G_j$, $j = 1, \ldots, l$, are finite dimensional complex vector spaces. Let $\Omega$ be a domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Consider the boundary value problem

$$\begin{cases} P(x,\partial)u = f, & \text{on} \quad \Omega, \\ B_j(x,\partial)u = g_j, & \text{on} \quad \partial\Omega, \quad j = 1,\ldots,l. \end{cases} \tag{3.1}$$

For fixed $x \in \partial\Omega$, define the half space $V_x := \{y \in \mathbb{R}^n; \langle y, \nu_x\rangle > 0\}$, where $\nu_x$ is the unit vector of inner normal to $\partial\Omega$ at point $x$. By a rotation if necessary, we can assume $n_x = (1,0,\ldots,0)$ and $P(x,\partial_x)$ can be written as

$$P(x,\partial) = \frac{\partial^m}{\partial x_1^m} + \sum_{\alpha=0}^{m-1} A_\alpha(x,\partial_{x'})\frac{\partial^\alpha}{\partial x_1^\alpha}, \tag{3.2}$$

up to multiply a invertible matrix, where order $A_\alpha(x,\partial_{x'}) = m-\alpha$, $x' = (x_2,\ldots,x_n)$. For the elliptic operator $P(x,\partial)$, the boundary value problem (1.6) is called *regular* if for any $\xi \in \mathbb{R}^{n-1}$ and $\eta_j \in G_j$, there is a unique bounded solution on $\mathbb{R}_+ = [0,\infty)$ to the Cauchy problem

$$\frac{d^m\Phi}{dt^m} + \sum_{\alpha=0}^{m-1} \widetilde{A}_\alpha(\xi)\frac{d^\alpha\Phi}{dt^\alpha} = 0, \quad \widetilde{B}_j\left(\xi, \frac{d}{dt}\right)\Phi(0) = \eta_j, \quad j = 1,\ldots,l, \tag{3.3}$$

Here $\Phi$ is a $E_0$-valued function over $\mathbb{R}_+$, $\widetilde{A}_\alpha(\xi)$ is the homogeneous part of $A_\alpha(x,\xi)$ of degree $m-\alpha$, and $A_\alpha(x,\xi)$ is obtained by replacing $\partial_{y'}$ in $A_\alpha(x,\partial_{y'})$ by $i\xi$ (this condition is the same if it is replaced by $\frac{1}{i}\xi$ ). The operator $\widetilde{B}_j(\xi, d/dt)$ is defined similarly. The regularity is equivalent to the fact that there is no nonzero bounded solution on $\mathbb{R}_+$ to the Cauchy problem

$$\frac{d^m\Phi}{dt^m} + \sum_{\alpha=0}^{m-1} \widetilde{A}_\alpha(\xi)\frac{d^\alpha\Phi}{dt^\alpha} = 0, \quad \widetilde{B}_j\left(\xi, \frac{d}{dt}\right)\Phi(0) = 0, \quad j = 1,\ldots,l. \tag{3.4}$$

Furthermore, it is equivalent to the fact that there is no nonzero rapidly decreasing solution on $\mathbb{R}_+$ to the Cauchy problem (3.4) (cf. ($ii'$) in p. 454 in [16]). This condition is usually called the *Lopatinski-Shapiro condition.*

The latter condition can also be stated without using rotations (cf. §20.1.1 in [12] and the discussion below it). For $x \in \partial\Omega$, and $\xi \perp \nu_x$, the map

$$M_{x,\xi} \ni u \longrightarrow (B_1(x, i\xi + \nu_x\partial_t)u(0), \ldots, B_l(x, i\xi + \nu_x\partial_t)u(0)) \tag{3.5}$$

is bijective, where $M_{x,\xi}$ is the set of all solutions $u \in C^\infty(\mathbb{R}_+, E_0)$ satisfying

$$P(x, i\xi + \nu_x\partial_t)u(t) = 0 \tag{3.6}$$

which are bounded on $\mathbb{R}_+$. Here for a differential operator $P$, the notation $P(\xi+\nu\partial_t)$ means that $\partial_{x_j}$ is replaced by $i\xi_j + \nu_j\partial_t$, $j = 1,\ldots,n$. Equivalently, there is no nonzero rapidly decreasing solution on $\mathbb{R}_+$ to the ODE (3.6) under the initial condition

$$B_j(x, i\xi + \nu_x\partial_t)u(0) = 0, \qquad j = 1,\ldots,l. \tag{3.7}$$

### 3.2. Checking the Shapiro-Lopatinskii condition for $k = 2$.

**Proposition 3.1.** *Suppose $\Omega$ is a smooth domain in $\mathbb{R}^4$. The boundary value problem*

$$\begin{cases} (D_0D_0^* + D_1^*D_1)\psi = 0, & \text{on} \quad \Omega, \\ D_0^*(\nu)\psi|_{\partial\Omega} = 0, & \\ D_1^*(\nu)D_1\psi|_{\partial\Omega} = 0, & \end{cases} \tag{3.8}$$

*is regular.*

*Proof.* Here we check the Lopatinski-Shapiro condition by generalizing the method proposed by Dain in [9], which we have used in [19]. Originally, this method works for operator of type $K^*K$ for some differential operator $K$ of first order, while here our operator has the form $D_0D_0^*+D_1^*D_1$.

Fix a point in the boundary $\partial\Omega$. Without loss of generality, we assume this point to be the origin. Denote by $\nu \in \mathbb{R}^4$ the unit vector of inner normal to the boundary at the origin. Let

$$\mathscr{V}_\nu = \{x \in \mathbb{R}^4; x \cdot \nu > 0\}$$

be a half-space. For any fixed vector $\xi \perp \nu$, suppose that $u(t)$ is a rapidly decreasing solution on $[0,\infty)$ to the following ODE under the initial condition:

$$\begin{cases} (D_0D_0^* + D_1^*D_1)(i\xi + \nu\partial_t)u(t) = 0, \\ D_0^*(\nu)u(0) = 0, \\ D_1^*(\nu)D_1(i\xi + \nu\partial_t)u(0) = 0. \end{cases} \tag{3.9}$$

Let us prove that $u$ vanishes. Now define a function $U : \mathscr{V}_\nu \to \mathbb{C}^4$ by

$$U(x) = e^{ix\cdot\xi}u(x \cdot \nu) \tag{3.10}$$

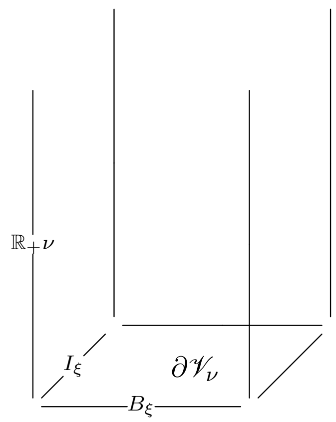
for $x \in \mathscr{V}_\nu$. Note that for a differential operator $Q = \sum_{j=0}^3 Q_j\partial_{x_j}$, where the $Q_j$'s are $(4 \times 3)$-matrices, we have $QU(x) = \sum_{j=0}^3 Q_j\left(i\xi_j u(x\cdot\nu) + \nu_j u'(x\cdot\nu)\right)e^{ix\cdot\xi}$. Then it is easy to see that (3.9) implies

$$\begin{cases} (D_0D_0^* + D_1^*D_1)U(x) = 0, \qquad \text{on} \quad \mathscr{V}_\nu, \\ D_0^*(\nu)U(x)|_{\partial\mathscr{V}_\nu} = 0, \\ D_1^*(\nu)D_1U(x)|_{\partial\mathscr{V}_\nu} = 0, \end{cases} \tag{3.11}$$

It is sufficient to show that $U$ vanishes. Consider the interval $I_\xi = \{s\xi \in \partial\mathscr{V}_\nu; |s| \le \frac{\pi}{|\xi|}\}$, the ball $B_\xi = \{y' \in \partial\mathscr{V}_\nu; y' \perp \xi, |y'| \le r\}$ for any fixed $r > 0$, and the domain

$$\mathscr{D}_\xi = I_\xi \times B_\xi \times \mathbb{R}_+\nu, \tag{3.12}$$

where $\mathbb{R}_+\nu = \{t\nu; t \in \mathbb{R}_+\}$.




Since $U$ in (3.10) rapidly decays in direction $\nu$, by Green's formula (2.16), we have

$$\begin{aligned} \int_{\mathscr{D}_\xi} \langle (D_0D_0^* + D_1^*D_1)U, U\rangle &= \int_{\mathscr{D}_\xi} \langle D_0^*U, D_0^*U\rangle + \int_{\mathscr{D}_\xi} \langle D_1U, D_1U\rangle \\ &- \int_{I_\xi\times B_\xi\times\{0\}\cup\partial I_\xi\times B_\xi\times\mathbb{R}_+\nu\cup I_\xi\times\partial B_\xi\times\mathbb{R}_+\nu} (\langle D_0^*U, D_0^*(\nu)U\rangle - \langle D_1^*(\nu)D_1U, U\rangle)dS, \end{aligned} \tag{3.13}$$

where $\langle\cdot,\cdot\rangle$ is the standard Hermitian inner product in $\mathbb{C}^4$.

(1) The integral $\int_{I_\xi\times B_\xi\times\{0\}}$ in (3.13) vanishes by the boundary condition $D_0^*(\nu)U=0$ and $D_1^*(\nu)D_1U=0$ on $\partial\mathscr{V}_\nu$ in (3.11);

(2) The integral $\int_{\partial I_\xi\times B_\xi\times\mathbb{R}_+\nu}$ vanishes since $U$, $D_0^*U$ and $D_1U$ are periodic in direction $\xi$, and on the opposite surface, we have the identity $D_j^*(\nu)|_{\{\xi\}\times B_\xi\times\mathbb{R}_+\nu}=-D_j^*(\nu)|_{\{-\xi\}\times B_\xi\times\mathbb{R}_+\nu}$, $j=0,1$;

(3) Similarly, the integral $\int_{I_\xi\times\partial B_\xi\times\mathbb{R}_+\nu}$ vanishes since $U$, $D_0^*U$ and $D_1U$ are constant in any direction in $B_\xi$, and on the opposite direction, we have the identity $D_j^*(\nu)|_{I_\xi\times\{v\}\times\mathbb{R}_+\nu}=-D_j^*(\nu)|_{I_\xi\times\{-v\}\times\mathbb{R}_+\nu}$ for any $v\in B_\xi$.

Obviously, the integral in the left hand side of (3.13) vanishes by the first equation in (3.11). Consequently,

$$\int_{\mathscr{D}_\xi}\langle D_0^*U,D_0^*U\rangle+\langle D_1U,D_1U\rangle=0,$$

i.e.,

$$D_0^*U=0,\qquad D_1U=0,\qquad \text{on}\quad \mathscr{V}_\nu. \tag{3.14}$$

By applying the following Proposition 3.2 to the convex domain $\mathscr{V}_\nu$, we see that there exists a function $\widetilde{U}\in C^\infty(\mathscr{V}_\nu,\mathbb{C}^3)$ such that $D_0\widetilde{U}=U$ on $\mathscr{V}_\nu$, and so $D_0^*D_0\widetilde{U}=0$ by the first identity in (3.14). By the explicit form of $D_0^*D_0$ in (2.19), we see that each component of $\widetilde{U}$ is harmonic on $\mathscr{V}_\nu$. Consequently, each component of $U=D_0\widetilde{U}$ is also harmonic on $\mathscr{V}_\nu$ since $\Delta U=\Delta D_0\widetilde{U}=D_0\Delta\widetilde{U}=0$ by $D_0$ being a differential operator of constant coefficients and $\triangle$ being a scalar differential operator of constant coefficients. This implies that

$$\begin{cases}\Delta U_0=\Delta U_1=\Delta U_2=\Delta U_3=0, & \text{on}\quad \mathscr{V}_\nu,\\ D_0^*(\nu)U|_{\partial\mathscr{V}_\nu}=0, & \\ D_1^*(\nu)D_1U|_{\partial\mathscr{V}_\nu}=0, & \end{cases} \tag{3.15}$$

In particular, when $\nu=(1,0,0,0)$, we have

$$\begin{cases}\Delta U_0=\Delta U_1=\Delta U_2=\Delta U_3=0, & \text{on}\quad \mathbb{R}^4_+,\\ U_1|_{\mathbb{R}^3}=U_2|_{\mathbb{R}^3}=0, & \\ (U_0-U_3)|_{\mathbb{R}^3}=0, & \\ \partial_{x_0}(U_0+U_3)|_{\mathbb{R}^3}=0, & \end{cases} \tag{3.16}$$

by the boundary conditions (2.17)-(2.18) for the upper half-space. Note that a harmonic function on $\mathbb{R}^4_+$ with vanishing boundary value must vanish. We see that $U_1\equiv U_2\equiv U_0-U_3\equiv 0$ and $\partial_{x_0}(U_0+U_3)\equiv 0$. Consequently, $U_0+U_3$ is independent of $x_0$, and so vanishes since it is rapidly decreasing in $x_0$. Therefore, $U\equiv 0$.

For the general case of $\nu$, we set

$$\zeta_0=\nu_0-i\nu_1,\qquad \zeta_1=\nu_2-i\nu_3. \tag{3.17}$$

Then

$$D_0(\nu)=\begin{pmatrix}-\overline{\zeta_1} & -\overline{\zeta_0} & 0\\ \zeta_0 & -\zeta_1 & 0\\ 0 & -\overline{\zeta_1} & -\overline{\zeta_0}\\ 0 & \zeta_0 & -\zeta_1\end{pmatrix}, \tag{3.18}$$

and

$$D_1(\nu)=(-\zeta_0,-\overline{\zeta_1},\zeta_1,-\overline{\zeta_0}). \tag{3.19}$$

It is direct to check that $D_1(\nu)D_0(\nu) = 0$. This also follows from $D_1D_0 = 0$. Note that

$$\det \begin{pmatrix} -\overline{\zeta_1} & -\overline{\zeta_0} \\ \zeta_0 & -\zeta_1 \end{pmatrix} = |\zeta_0|^2 + |\zeta_1|^2, \tag{3.20}$$

and therefore $D_0(\nu)$ in (3.18) has rank 3. The vector $D_1(\nu)$ in (3.19) does not vanish for nonvanishing $\nu$, i.e., $D_1(\nu)$ has rank 1. Hence, $\mathrm{Im} D_0(\nu) = \ker D_1(\nu)$ and $\mathrm{Im} D_1(\nu)^*$ is a 1-dimensional space orthogonal to $\ker D_1(\nu)$. Namely we have the orthogonal decomposition

$$\mathbb{C}^4 = \mathrm{Im} D_0(\nu) \oplus \mathrm{Im} D_1(\nu)^*.$$

We rewrite $U$ as

$$U = D_0(\nu)U' + D_1(\nu)^*U'',$$

for some $\mathbb{C}^3$-valued function $U'$ and scalar function $U''$. Then,

$$D_0^*(\nu)U = D_0^*(\nu)(D_0(\nu)U' + D_1(\nu)^*U'') = D_0^*(\nu)D_0(\nu)U'. \tag{3.21}$$

Here $D_0^*(\nu)D_0(\nu)$ is an invertible $(3 \times 3)$-matrix because $D_0(\nu)$ has rank 3. It follows from $D_0^*(\nu)D_0(\nu)\triangle U' = D_0^*(\nu)\triangle U = 0$ that $U'$ is harmonic. So is $U''$ if we choose $U''(x)$ orthogonal to $\ker D_1(\nu)^*$ at each point $x$. The second equation in (3.15) together with (3.21) implies that $U' = 0$ on the boundary $\partial\mathscr{V}_\nu$, and so it vanishes as a harmonic function on the whole half space $\mathscr{V}_\nu$. Now we have $U = D_1(\nu)^*U''$.

The third equation in (3.15) implies that the scalar function $D_1U|_{\partial\mathscr{V}_\nu} = 0$. Then,

$$\begin{aligned} D_1U &= D_1D_1(\nu)^*U'' = (-\partial_{z_0}, -\partial_{\bar{z}_1}, \partial_{z_1}, -\partial_{\bar{z}_0})\, D_1(\nu)^*U'' \\ &= (-(\partial_{x_0} - i\partial_{x_1}), -(\partial_{x_2} + i\partial_{x_3}), \partial_{x_2} - i\partial_{x_3}, -(\partial_{x_0} + i\partial_{x_1})) \begin{pmatrix} -(\nu_0 + i\nu_1) \\ -(\nu_2 - i\nu_3) \\ \nu_2 + i\nu_3 \\ -(\nu_0 - i\nu_1) \end{pmatrix} U'' \\ &= 2(\nu_0\partial_{x_0} + \nu_1\partial_{x_1} + \nu_2\partial_{x_2} + \nu_3\partial_{x_3})U'' = 2\partial_\nu U'' = 0 \end{aligned} \tag{3.22}$$

on the boundary $\partial\mathscr{V}_\nu$. As a harmonic function, we must have $\partial_\nu U'' \equiv 0$ on the whole half space $\mathscr{V}_\nu$. So $U''$ is constant in the direction $\nu$. But it is also rapidly decreasing along this direction. Hence $U'' \equiv 0$ on $\mathscr{V}_\nu$. Thus $U$ vanishes on $\mathscr{V}_\nu$. □

3.3. **The solvability of the non-homogeneous $k$-Cauchy-Fueter equations on convex domains without estimate.** The following proposition is proved in [20] for any dimension by using twistor transformations. Here we give an elementary proof.

**Proposition 3.2.** *The sequence*

$$C^\infty(\Omega, \mathbb{C}^3) \xrightarrow{D_0} C^\infty(\Omega, \mathbb{C}^4) \xrightarrow{D_1} C^\infty(\Omega, \mathbb{C}^1), \tag{3.23}$$

*is exact for any convex domain $\Omega$. Namely, for any $\psi \in C^\infty(\Omega, \mathbb{C}^4)$ satisfying $D_1\psi = 0$, there exists $\phi \in C^\infty(\Omega, \mathbb{C}^3)$ such that*

$$D_0\phi = \psi \qquad \text{on} \quad \Omega.$$

Let $\mathscr{E}(\Omega)$ be the set of all $C^\infty$ function on $\Omega$ and let $\mathscr{R}$ be the ring of polynomials $\mathbb{C}[\xi_0, \xi_1, \ldots, \xi_n]$. For a positive integer $p$, $\mathscr{R}^p$ denotes the space of all vectors $(f_1, \ldots, f_p)^t$ with $f_1, \ldots, f_p \in \mathscr{R}$, and $\mathscr{E}^p(\Omega)$ is defined similarly. The following result is essentially due to Ehrenpreis-Malgrange-Palamodov.

**Theorem 3.1.** (cf. Theorem A in [15]) *Let $A(\xi)$, $B(\xi)$ be respectively $(q\times p)$ and $(r\times q)$ matrices of polynomials, and let $A(D)$ and $B(D)$ be differential operators obtained by substituting $\partial_{x_j}$ to $\frac{1}{i}\xi_j$ to $A(\xi)$ and $B(\xi)$, respectively. Then the following statements are equivalent.*

*(1) the sequence $\mathscr{R}^p \xleftarrow{A(\xi)^t} \mathscr{R}^q \xleftarrow{B(\xi)^t} \mathscr{R}^r$ is exact.*

*(2) the sequence $\mathscr{E}^p(\Omega) \xrightarrow{A(D)} \mathscr{E}^q(\Omega) \xrightarrow{B(D)} \mathscr{E}^r(\Omega)$ is exact for any domain $\Omega \subset \mathbb{R}^{n+1}$ convex and non empty.*

**Lemma 3.1.** *The sequence*

$$0 \leftarrow \mathbb{C}^3 \xleftarrow{D_0(\xi)^t} \mathbb{C}^4 \xleftarrow{D_1(\xi)^t} \mathbb{C}^1 \leftarrow 0$$

*is exact for any nonzero $\xi \in \mathbb{C}^4$.*

*Proof.* Set

$$\eta_0 = \xi_0 - i\xi_1, \qquad \eta_1 = \xi_2 - i\xi_3 \tag{3.24}$$

Then

$$D_0(\xi)^t = \frac{1}{i}\begin{pmatrix} -\overline{\eta_1} & \eta_0 & 0 & 0 \\ -\overline{\eta_0} & -\eta_1 & -\overline{\eta_1} & \eta_0 \\ 0 & 0 & -\overline{\eta_0} & -\eta_1 \end{pmatrix}, \tag{3.25}$$

and

$$D_1(\xi)^t = \frac{1}{i}\begin{pmatrix} -\eta_0 \\ -\overline{\eta_1} \\ \eta_1 \\ -\overline{\eta_0} \end{pmatrix}. \tag{3.26}$$

The proof of $\mathrm{Im} D_1(\xi)^t = \ker D_0(\xi)^t$ is similar to the paragraph below (3.20). □

**Proposition 3.3.** *The sequence $\mathscr{R}^3 \xleftarrow{D_0(\xi)^t} \mathscr{R}^4 \xleftarrow{D_1(\xi)^t} \mathscr{R}^1$ is exact.*

*Proof.* It is obvious that $D_0(\xi)^t D_1(\xi)^t = 0$ by (3.25)-(3.25). Suppose $D_0(\xi)^t \begin{pmatrix} p_1(\xi) \\ \vdots \\ p_4(\xi) \end{pmatrix} = 0$, where $p_j$ are polynomials. By Lemma 3.1, for each $\xi \neq 0$, there exists an element of $\mathbb{C}^1$, say $f_\xi$, such that

$$\begin{pmatrix} p_1(\xi) \\ \vdots \\ p_4(\xi) \end{pmatrix} = D_1(\xi)^t f_\xi = \frac{1}{i}\begin{pmatrix} -\xi_0 + i\xi_1 \\ -\xi_2 - i\xi_3 \\ \xi_2 - i\xi_3 \\ -\xi_0 - i\xi_1 \end{pmatrix} f_\xi.$$

It follows form the first two equations that $(\xi_0 + i\xi_1)p_1(\xi) + (\xi_2 - i\xi_3)p_2(\xi) = i(\xi_1^2 + \ldots + \xi_4^2)f_\xi$ on $\mathbb{R}^4 \setminus \{0\}$. Then $f_\xi$ is a rational function $Q(\xi)/(\xi_1^2 + \ldots + \xi_4^2)$ for some polynomial $Q(\xi)$. The first equation above implies the following identity of polynomials:

$$ip_1(\xi)(\xi_1^2 + \ldots + \xi_4^2) = (-\xi_0 + i\xi_1)Q(\xi).$$

This equation also holds on $\mathbb{C}^4$ by natural extension of polynomials. By comparison of zero locuses, we see that $-\xi_0 + i\xi_1$ must be a factor of $p_1(\xi)$. Namely, $p_1(\xi) = (-\xi_0 + i\xi_1)q(\xi)$ for some polynomial $q(\xi)$. Consequently, $f_\xi = iq(\xi)$ is a polynomial on $\mathbb{R}^4$. The result follows. □

Applying Theorem 3.1 to the exact sequence in Proposition 3.3, we get the Proposition 3.2.

## 4. The case $k > 2$

4.1. **The operators $D_0^{(k)}$ and $D_1^{(k)}$ and the associated Laplacian.** The operators in the $k$-Cauchy-Fueter complex (2.3) are given by (2.4). If we use notations

$$\phi = \begin{pmatrix} \phi_{0'0'\ldots 0'0'} \\ \phi_{1'0'\ldots 0'0'} \\ \vdots \\ \phi_{1'1'\ldots 1'0'} \\ \phi_{1'1'\ldots 1'1'} \end{pmatrix} = \begin{pmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_k \end{pmatrix}, \qquad \psi := \begin{pmatrix} \psi_{00'\ldots 0'0'} \\ \psi_{10'\ldots 0'0'} \\ \vdots \\ \psi_{01'\ldots 1'1'} \\ \psi_{11'\ldots 1'1'} \end{pmatrix} = \begin{pmatrix} \vdots \\ \psi_{0,j} \\ \psi_{1,j} \\ \vdots \\ \psi_{0,k-1} \\ \psi_{1,k-1} \end{pmatrix} = \begin{pmatrix} \vdots \\ \psi_{2j} \\ \psi_{2j+1} \\ \vdots \\ \psi_{2k-2} \\ \psi_{2k-1} \end{pmatrix}, \tag{4.1}$$

where $\phi_j := \phi_{1'\ldots 1'0'\ldots 0'}$ with $j$ indices equal $1'$, $\psi_{A,j} := \psi_{A,1'\ldots 1'0'\ldots 0'}$ with $j$ indices equal $1'$, $A = 0, 1$, then the operator $D_0^{(k)}$ in (2.4) can be written as a $(2k) \times (k+1)$-matrix valued differential operator of first order from $C^1(\Omega, \mathbb{C}^{k+1})$ to $C^0(\Omega, \mathbb{C}^{2k})$ as follows

$$D_0^{(k)} = \begin{pmatrix} -\partial_{\overline{z}_1} & -\partial_{\overline{z}_0} & 0 & 0 & 0 & \cdots \\ \partial_{z_0} & -\partial_{z_1} & 0 & 0 & 0 & \cdots \\ 0 & -\partial_{\overline{z}_1} & -\partial_{\overline{z}_0} & 0 & 0 & \cdots \\ 0 & \partial_{z_0} & -\partial_{z_1} & 0 & 0 & \cdots \\ 0 & 0 & -\partial_{\overline{z}_1} & -\partial_{\overline{z}_0} & 0 & \cdots \\ 0 & 0 & \partial_{z_0} & -\partial_{z_1} & 0 & \cdots \\ 0 & 0 & 0 & -\partial_{\overline{z}_1} & -\partial_{\overline{z}_0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

(cf. [20]) and so

$$D_0^{(k)*} = -\begin{pmatrix} -\partial_{z_1} & \partial_{\overline{z}_0} & 0 & 0 & 0 & 0 & 0 & \cdots \\ -\partial_{z_0} & -\partial_{\overline{z}_1} & -\partial_{z_1} & \partial_{\overline{z}_0} & 0 & 0 & 0 & \cdots \\ 0 & 0 & -\partial_{z_0} & -\partial_{\overline{z}_1} & -\partial_{z_1} & \partial_{\overline{z}_0} & 0 & \cdots \\ 0 & 0 & 0 & 0 & -\partial_{z_0} & -\partial_{\overline{z}_1} & -\partial_{z_1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}. \tag{4.2}$$

Then

$$D_0^{(k)} D_0^{(k)*} = -\begin{pmatrix} \triangle & 0 & \partial_{\overline{z}_0}\partial_{z_1} & -\partial_{\overline{z}_0}^2 & 0 & 0 & 0 & \cdots \\ * & \triangle & \partial_{z_1}^2 & -\partial_{\overline{z}_0}\partial_{z_1} & 0 & 0 & 0 & \cdots \\ * & * & \triangle & 0 & \partial_{\overline{z}_0}\partial_{z_1} & -\partial_{\overline{z}_0}^2 & 0 & \cdots \\ * & * & * & \triangle & \partial_{z_1}^2 & -\partial_{\overline{z}_0}\partial_{z_1} & 0 & \cdots \\ * & * & * & * & \triangle & 0 & \partial_{\overline{z}_0}\partial_{z_1} & \cdots \\ * & * & * & * & * & \triangle & \partial_{z_1}^2 & \cdots \\ * & * & * & * & * & * & \triangle & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \tag{4.3}$$

by direct calculation. Here $*$-entries are known again by Hermitian symmetry.

By Green's formula (2.16), $\psi \in \mathrm{Dom} D_0^{(k)*} \cap C^1(\Omega, \mathbb{C}^{2k})$ if and only if $D_0^{(k)*}(\nu)\psi = 0$ on the boundary. When $\nu = (1, 0, 0, 0)$ this condition becomes

$$0 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 0 \end{pmatrix} \psi|_{\partial\Omega},$$

from which we get

$$\psi_1 = \psi_{2k-2} = 0, \qquad \psi_j - \psi_{j+3} = 0, \qquad j = 0, 2, 4, \ldots, 2k-4. \tag{4.4}$$

A section $\Psi \in C^\infty(\Omega, \odot^{k-2}\mathbb{C}^2 \otimes \Lambda^2\mathbb{C}^2)$ has $(k-1)$ components $\Psi_{010'\ldots0'}, \Psi_{011'\ldots0'}, \ldots, \Psi_{011'\ldots1'}$. We use notations $\Psi_j := \Psi_{011'\ldots1'0'\ldots0'}$ with $j$ indices equal $1'$ and $\psi_j$ as in (4.1). By definition $(D_1^{(k)}\psi)_{01B'\ldots C'} = \sum_{A'=0',1'} \left(\nabla_0^{A'}\psi_{1A'B'\ldots C'} - \nabla_1^{A'}\psi_{0A'B'\ldots C'}\right)$, and we have

$$\left(D_1^{(k)}\psi\right)_j = -\nabla_1^{0'}\psi_{0,j} + \nabla_0^{0'}\psi_{1,j} - \nabla_1^{1'}\psi_{0,j+1} + \nabla_0^{1'}\psi_{1,j+1},$$

$j = 0, \ldots, k-2$. We see that $\Psi = D_1^{(k)}\psi$ with

$$\begin{aligned} D_1^{(k)} &= \begin{pmatrix} -\nabla_1^{0'} & \nabla_0^{0'} & -\nabla_1^{1'} & \nabla_0^{1'} & 0 & 0 & 0 & \cdots \\ 0 & 0 & -\nabla_1^{0'} & \nabla_0^{0'} & -\nabla_1^{1'} & \nabla_0^{1'} & 0 & \cdots \\ 0 & 0 & 0 & 0 & -\nabla_1^{0'} & \nabla_0^{0'} & -\nabla_1^{1'} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \\ &= \begin{pmatrix} -\partial_{z_0} & -\partial_{\overline{z}_1} & \partial_{z_1} & -\partial_{\overline{z}_0} & 0 & 0 & 0 & \cdots \\ 0 & 0 & -\partial_{z_0} & -\partial_{\overline{z}_1} & \partial_{z_1} & -\partial_{\overline{z}_0} & 0 & \cdots \\ 0 & 0 & 0 & 0 & -\partial_{z_0} & -\partial_{\overline{z}_1} & \partial_{z_1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \end{aligned} \tag{4.5}$$

a $(k-1) \times (2k)$-matrix operator, and

$$D_1^{(k)*} = -\begin{pmatrix} -\partial_{\overline{z}_0} & 0 & 0 & \cdots \\ -\partial_{z_1} & 0 & 0 & \cdots \\ \partial_{\overline{z}_1} & -\partial_{\overline{z}_0} & 0 & \cdots \\ -\partial_{z_0} & -\partial_{z_1} & 0 & \cdots \\ 0 & \partial_{\overline{z}_1} & -\partial_{\overline{z}_0} & \cdots \\ 0 & -\partial_{z_0} & -\partial_{z_1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{4.6}$$

a $(2k) \times (k-1)$-matrix operator. So

$$D_1^{(k)*}D_1^{(k)} = -\begin{pmatrix} \partial_{z_0}\partial_{\overline{z}_0} & \partial_{\overline{z}_0}\partial_{\overline{z}_1} & -\partial_{\overline{z}_0}\partial_{z_1} & \partial_{\overline{z}_0}^2 & 0 & 0 & \cdots \\ * & \partial_{\overline{z}_1}\partial_{z_1} & -\partial_{z_1}^2 & \partial_{\overline{z}_0}\partial_{z_1} & 0 & 0 & \cdots \\ * & * & \triangle & 0 & -\partial_{\overline{z}_0}\partial_{z_1} & \partial_{\overline{z}_0}^2 & \cdots \\ * & * & * & \triangle & -\partial_{z_1}^2 & \partial_{\overline{z}_0}\partial_{z_1} & \cdots \\ * & * & * & * & \triangle & 0 & \cdots \\ * & * & * & * & * & \triangle & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}. \tag{4.7}$$

The sum of (4.3) and (4.7) gives

(4.8)
$$\Box_1^{(k)} := D_0^{(k)} D_0^{(k)*} + D_1^{(k)*} D_1^{(k)} = -\begin{pmatrix} \Delta+\Delta_1 & L & 0 & \cdots & 0 & 0 & 0 \\ \overline{L} & \Delta+\Delta_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 2\triangle & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2\triangle & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \Delta+\Delta_2 & -L \\ 0 & 0 & 0 & \cdots & 0 & -\overline{L} & \Delta+\Delta_1 \end{pmatrix}.$$

This is an elliptic operator. Using the notation in (3.17), we obtain

$$D_0^{(k)}(\nu) = \begin{pmatrix} -\overline{\zeta_1} & -\overline{\zeta_0} & 0 & 0 & 0 & \cdots \\ \zeta_0 & -\zeta_1 & 0 & 0 & 0 & \cdots \\ 0 & -\overline{\zeta_1} & -\overline{\zeta_0} & 0 & 0 & \cdots \\ 0 & \zeta_0 & -\zeta_1 & 0 & 0 & \cdots \\ 0 & 0 & -\overline{\zeta_1} & -\overline{\zeta_0} & 0 & \cdots \\ 0 & 0 & \zeta_0 & -\zeta_1 & 0 & \cdots \\ 0 & 0 & 0 & -\overline{\zeta_1} & -\overline{\zeta_0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{4.9}$$

a $(2k) \times (k+1)$-matrix, and

$$D_1^{(k)}(\nu) = \begin{pmatrix} -\zeta_0 & -\overline{\zeta_1} & \zeta_1 & -\overline{\zeta_0} & 0 & 0 & 0 & \cdots \\ 0 & 0 & -\zeta_0 & -\overline{\zeta_1} & \zeta_1 & -\overline{\zeta_0} & 0 & \cdots \\ 0 & 0 & 0 & 0 & -\zeta_0 & -\overline{\zeta_1} & \zeta_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \tag{4.10}$$

is a $(k-1) \times (2k)$-matrix. And a $(2k) \times (k-1)$-matrix

$$D_1^{(k)*}(\nu) = -\begin{pmatrix} -\overline{\zeta_0} & 0 & 0 & 0 & \cdots \\ -\zeta_1 & 0 & 0 & 0 & \cdots \\ \overline{\zeta_1} & -\overline{\zeta_0} & 0 & 0 & \cdots \\ -\zeta_0 & -\zeta_1 & 0 & 0 & \cdots \\ 0 & \overline{\zeta_1} & 0 & 0 & \cdots \\ 0 & -\zeta_0 & -\overline{\zeta_0} & 0 & \cdots \\ 0 & 0 & \zeta_1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{4.11}$$

By Green's formula (2.16), $\Psi \in \mathrm{Dom} D_1^{(k)*} \cap C^1(\Omega, \mathbb{C}^{k-1})$ if and only if $D_1^{(k)*}(\nu)\Psi|_{\partial\Omega} = 0$ on the boundary. It follows from $D_1^{(k)*}(\nu)$ in (4.11) that $\Psi|_{\partial\Omega} = 0$ since $\zeta_0$ and $\zeta_1$ can not vanish simultaneously. Now $D_1^{(k)}\psi \in \mathrm{Dom} D_1^{(k)*} \cap C^1(\Omega, \mathbb{C}^{k-1})$ if and only if $D_1^{(k)}\psi = 0$ on the boundary. So (1.6) is our natural boundary value condition.

4.2. **The boundary value problem (1.6) satisfies the Shapiro-Lopatinskii condition.** Suppose $u(t)$ is a rapidly decreasing solution on $[0, \infty)$ to the following ODE under the initial condition:

$$
\begin{cases} (D_0^{(k)}D_0^{(k)*}+D_1^{(k)*}D_1^{(k)})(i\xi+\nu\partial_t)u(t)=0, \\ D_0^{(k)*}(\nu)u(0)=0, \\ D_1^{(k)*}(\nu)D_1^{(k)}(i\xi+\nu\partial_t)u(0)=0. \end{cases} \tag{4.12}
$$

Define a function $U:\mathscr{V}_\nu\to\mathbb{C}^{2k}$ as in (3.10). Now let us show $U$ vanishing. Then the boundary value problem (4.12) satisfies the Lopatinski-Shapiro condition. By the argument as in the case $k=2$ and using the following Proposition 4.1, we find that

$$
\begin{cases} \Delta U=0, \qquad \text{on} \quad \mathscr{V}_\nu, \\ D_0^{(k)*}(\nu)U|_{\partial\mathscr{V}_\nu}=0, \\ D_1^{(k)*}(\nu)D_1^{(k)}U|_{\partial\mathscr{V}_\nu}=0, \end{cases} \tag{4.13}
$$

It is direct to check that $D_1^{(k)}(\nu)D_0^{(k)}(\nu)=0$, which can be also obtained from $D_1^{(k)}D_0^{(k)}=0$. The matrix $D_0^{(k)}(\nu)$ in (4.9) has rank $k+1$, and $D_1^{(k)}(\nu)$ in (4.10) has rank $k-1$. Moreover, $\mathrm{Im}D_0^{(k)}(\nu)=\ker D_1^{(k)}(\nu)$ and the space $\mathrm{Im}D_1^{(k)*}(\nu)$ is $(k-1)$-dimensional, orthogonal to $\ker D_1^{(k)}(\nu)$. Namely we have the orthogonal decomposition

$$
\mathbb{C}^{2k}=\mathrm{Im}D_0^{(k)}(\nu)\oplus\mathrm{Im}D_1^{(k)*}(\nu)\cong\mathbb{C}^{k+1}\oplus\mathbb{C}^{k-1},
$$

(cf. (2.13) in [18]). We rewrite $U$ as

$$
U=D_0^{(k)}(\nu)U'+D_1^{(k)*}(\nu)U'',
$$

for some $\mathbb{C}^{k+1}$-valued function $U'$ and $\mathbb{C}^{k-1}$-valued function $U''$. Then,

$$
D_0^{(k)*}(\nu)U=D_0^{(k)*}(\nu)D_0^{(k)}(\nu)U'.
$$

Here $D_0^{(k)*}(\nu)D_0^{(k)}(\nu)$ is an invertible $(k+1)\times(k+1)$-matrix because $D_0^{(k)}(\nu)$ has rank $k+1$. Consequently, $U''$ and $U'$ are both harmonic. The second equation in (3.15) implies that $U'=0$ on the boundary $\partial\mathscr{V}_\nu$, and so it vanishes as a harmonic function on the whole half space $\mathscr{V}_\nu$. Now we have $U=D_1^{(k)*}(\nu)U''$.

Now the third equation in (4.13) implies that $D_1^{(k)}U|_{\partial\mathscr{V}_\nu}=0$ by $D_1^{(k)*}(\nu)$ in (4.11). Note that

$$
(-\partial_{z_0},-\partial_{\overline{z}_1},\partial_{z_1},-\partial_{\overline{z}_0})\begin{pmatrix} -\overline{\zeta_0} \\ -\zeta_1 \\ \overline{\zeta_1} \\ -\zeta_0 \end{pmatrix}=2\partial_\nu
$$

as in (3.22), and

$$
\mathscr{L}:=(-\partial_{z_0},-\partial_{\overline{z}_1})\begin{pmatrix} \overline{\zeta_1} \\ -\zeta_0 \end{pmatrix}=-(\partial_{x_0}-i\partial_{x_1})(\nu_2+i\nu_3)+(\partial_{x_2}+i\partial_{x_3})(\nu_0-i\nu_1)=\partial_\mu+i\partial_{\widetilde{\mu}},
$$

where

$$
\mu=(-\nu_2,-\nu_3,\nu_0,\nu_1), \qquad \widetilde{\mu}=(-\nu_3,\nu_2,-\nu_1,\nu_0), \tag{4.14}
$$

and

$$
(\partial_{z_1},-\partial_{\overline{z}_0})\begin{pmatrix} -\overline{\zeta_0} \\ -\zeta_1 \end{pmatrix}=-\overline{\mathscr{L}}.
$$

Then we find that

(4.15)
$$D_1^{(k)}U = D_1^{(k)}D_1^{(k)*}(\nu)U'' = \begin{pmatrix} 2\partial_\nu & -\overline{\mathscr{L}} & 0 & \cdots & 0 & 0 & 0 \\ \mathscr{L} & 2\partial_\nu & -\overline{\mathscr{L}} & \cdots & 0 & 0 & 0 \\ 0 & \mathscr{L} & 2\partial_\nu & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mathscr{L} & 2\partial_\nu & -\overline{\mathscr{L}} \\ 0 & 0 & 0 & \cdots & 0 & \mathscr{L} & 2\partial_\nu \end{pmatrix} \begin{pmatrix} U_1'' \\ \vdots \\ U_{k-1}'' \end{pmatrix} = 0$$

on the boundary $\partial\mathscr{V}_\nu$, by using $D_1^{(k)}$ in (4.5) and $D_1^{(k)*}(\nu)$ in (4.11).

When $k = 3$, we obtain that

$$\begin{cases} 2\partial_\nu U_1'' - (\partial_\mu - i\partial_{\widetilde{\mu}})U_2'' = 0, \\ (\partial_\mu + i\partial_{\widetilde{\mu}})U_1'' + 2\partial_\nu U_2'' = 0, \end{cases} \tag{4.16}$$

on the boundary $\partial\mathscr{V}_\nu$. Both $2\partial_\nu U_1'' - (\partial_\mu - i\partial_{\widetilde{\mu}})U_2''$ and $(\partial_\mu + i\partial_{\widetilde{\mu}})U_1'' + 2\partial_\nu U_2''$ are harmonic functions on $\mathscr{V}_\nu$, and so must vanish. Namely, (4.16) holds on the whole half space $\mathscr{V}_\nu$. On the other hand, as a harmonic function, $\triangle U = e^{ix\cdot\xi}(u'' - |\xi|^2 u)(x\cdot\nu) = 0$. So as a rapidly decreasing function, we must have $u(t) = e^{-|\xi|t}u_0$ for some vector $u_0 \in \mathbb{C}^6$. Consequently, $U'' = e^{ix\cdot\xi - |\xi|x\cdot\nu}W''$ for some vector $W'' \in \mathbb{C}^2$ Then substitute it into (4.16) to get

$$\begin{pmatrix} -2|\xi| & \overline{\Lambda} \\ \Lambda & -2|\xi| \end{pmatrix} \begin{pmatrix} W_1'' \\ W_2'' \end{pmatrix} = 0$$

where $\Lambda = i(\mu\cdot\xi + i\widetilde{\mu}\cdot\xi)$.

$$\det\begin{pmatrix} -2|\xi| & \overline{\Lambda} \\ \Lambda & -2|\xi| \end{pmatrix} = 4|\xi|^2 - |\Lambda|^2 > 0,$$

by $|\Lambda| \leq |\xi|$ since $\mu$ and $\widetilde{\mu}$ are mutually orthogonal unit vectors in the hyperplane orthogonal to $\nu$ (cf. (4.14)), and $\xi \perp \nu$. Hence $W'' = 0$ and $U$ vanishes.

In the case $k > 3$, $U'' = e^{ix\cdot\xi - |\xi|x\cdot\nu}W''$ for some vector $W'' \in \mathbb{C}^{k-1}$ Then substitute it into (4.17), which also holds on the whole half space $\mathscr{V}_\nu$, to get

$$\begin{pmatrix} -2|\xi| & \overline{\Lambda} & 0 & 0 & 0 & 0 & 0 & \cdots \\ \Lambda & -2|\xi| & \overline{\Lambda} & 0 & 0 & 0 & 0 & \cdots \\ 0 & \Lambda & -2|\xi| & \overline{\Lambda} & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} W'' = 0. \tag{4.17}$$

It sufficient to show the determinant of the above matrix vanishing. This is true because the determinant equals to

$$\det\begin{pmatrix} -2|\xi| & \overline{\Lambda} & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \lambda' & \overline{\Lambda} & 0 & 0 & 0 & 0 & \cdots \\ 0 & \Lambda & -2|\xi| & \overline{\Lambda} & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} = \det\begin{pmatrix} -2|\xi| & \overline{\Lambda} & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \lambda' & \overline{\Lambda} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \lambda'' & \overline{\Lambda} & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

with $\lambda' = -|\xi|(2 - \frac{|\Lambda|^2}{2|\xi|^2}) < -|\xi|$. Then $\lambda'' = -|\xi|(2 - \frac{|\Lambda|^2}{|\lambda'||\xi|}) < -|\xi|$ if $\lambda' < -|\xi|$. Repeating this procedure, we see that the above determinant is nonzero. So $U$ vanishes and we complete the proof of the regularity of the boundary value problem (4.12).

**Lemma 4.1.** *For any nonzero $\xi \in \mathbb{R}^4$, the sequence*

$$0 \leftarrow \mathbb{C}^{k+1} \xleftarrow{D_0^{(k)}(\xi)^t} \mathbb{C}^{2k} \xleftarrow{D_1^{(k)}(\xi)^t} \mathbb{C}^{k-1} \leftarrow 0$$

*is exact.*

*Proof.* With $\eta$ as in (3.24),

$$D_0^{(k)}(\xi)^t = \frac{1}{i}\begin{pmatrix} -\overline{\eta}_1 & \eta_0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -\overline{\eta}_0 & -\eta_1 & -\overline{\eta_1} & \eta_0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & -\overline{\eta}_0 & -\eta_1 & -\overline{\eta_1} & \eta_0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & -\overline{\eta}_0 & -\eta_1 & -\overline{\eta}_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$D_1^{(k)}(\xi)^t = \frac{1}{i}\begin{pmatrix} -\eta_0 & 0 & 0 & 0 & \cdots \\ -\overline{\eta_1} & 0 & 0 & 0 & \cdots \\ \eta_1 & -\eta_0 & 0 & 0 & \cdots \\ -\overline{\eta_0} & -\overline{\eta_1} & 0 & 0 & \cdots \\ 0 & \eta_1 & 0 & 0 & \cdots \\ 0 & -\overline{\eta_0} & -\eta_0 & 0 & \cdots \\ 0 & 0 & \overline{\eta_1} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The proof of the equality $\mathrm{Im} D_1^{(k)}(\xi)^t = \ker D_0^{(k)}(\xi)^t$ follows as in the case of $k = 2$. □

**Proposition 4.1.** *The sequence $\mathscr{R}^{k+1} \xleftarrow{D_0^{(k)}(\xi)^t} \mathscr{R}^{2k} \xleftarrow{D_1^{(k)}(\xi)^t} \mathscr{R}^{k-1}$ is exact.*

*Proof.* Suppose $D_0^{(k)}(\xi)^t \begin{pmatrix} p_1(\xi) \\ \vdots \\ p_{2k}(\xi) \end{pmatrix} = 0$, where $p_j$ are polynomials. For each $\xi \neq 0$, there exists a unique $f_\xi = (f_{\xi;1}, \ldots, f_{\xi;k-1})^t \in \mathbb{C}^{k-1}$, such that

$$\begin{pmatrix} p_1(\xi) \\ \vdots \\ p_{2k}(\xi) \end{pmatrix} = D_1^{(k)}(\xi)^t f_\xi = \frac{1}{i}\begin{pmatrix} -\xi_0 + i\xi_1 & 0 & 0 & 0 & \cdots \\ -\xi_2 - i\xi_3 & 0 & 0 & 0 & \cdots \\ \xi_2 - i\xi_3 & -\xi_0 + i\xi_1 & 0 & 0 & \cdots \\ -\xi_0 - i\xi_1 & -\xi_2 - i\xi_3 & 0 & 0 & \cdots \\ 0 & \xi_2 - i\xi_3 & 0 & 0 & \cdots \\ 0 & -\xi_0 - i\xi_1 & -\xi_0 + i\xi_1 & 0 & \cdots \\ 0 & 0 & -\xi_2 - i\xi_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} f_{\xi;1} \\ \vdots \\ f_{\xi;k-1} \end{pmatrix}.$$

In the same way as in the case $k = 2$, we can show that $f_{\xi;1}$ is a polynomial. Then repeat this procedure for $f_{\xi;2}, f_{\xi;3}, \ldots$. □

## 5. Proofs of main theorems

5.1. **The case $\Box_2^{(k)}$.** It is direct to see that

$$\Box_2^{(k)} = D_1^{(k)} D_1^{(k)*} = \begin{pmatrix} 2\triangle & 0 & \cdots \\ 0 & 2\triangle & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix},$$

by (4.5)-(4.6). The condition $D_1^{(k)*}(\nu)\Psi = 0$ on the boundary $\partial\Omega$ implies that $\Psi|_{\partial\Omega} = 0$ as before. The boundary value problem

$$\begin{cases} \square_2^{(k)}\Psi = 0, & \text{on} \quad \Omega, \\ \Psi|_{\partial\Omega} = 0, \end{cases} \tag{5.1}$$

is just the boundary value problem for the usual Laplacian operator with Dirichlet boundary value. It is always solvable by the solution operator $N_2^{(k)} : H^s(\Omega, \mathbb{C}^{k-1}) \to H^{s+2}(\Omega, \mathbb{C}^{k-1})$. Consequently, we have

$$\Psi = D_1^{(k)} D_1^{(k)*} N_2^{(k)} \Psi$$

and the equation

$$D_1^{(k)}\psi = \Psi,$$

is uniquely solved by $u = D_1^{(k)*} N_2^{(k)}\Psi$ for any $\Psi \in H^s(\Omega, \mathbb{C}^{k-1})$.

5.2. **The Fredholm property.**

**Theorem 5.1.** *(Proposition 11.14 and 11.16 in* [16], *Theorem 20.1.8 in* [12]*) Suppose that the boundary value problem (3.1) is regular. Then the operator*

$$T : H^{m+s}(\Omega, E_0) \longrightarrow H^m(\Omega, E_1) \oplus \bigoplus_{j=1}^{l} H^{m+s-m_j-\frac{1}{2}}(\partial\Omega, G_j),$$

$s = 0, 1, \ldots,$ *defined by*

$$Tu = (P(x,\partial)u, B_1(x,\partial)u, \ldots, B_l(x,\partial)u)$$

*is Fredholm, and satisfies the estimate*

$$\|u\|^2_{H^{m+s}(\Omega)} \le C \left( \|Pu\|^2_{H^s(\Omega)} + \sum_{j=1}^{l} \|B_j u\|^2_{H^{m+s-m_j-\frac{1}{2}}(\partial\Omega)} + \|u\|^2_{H^{m-1}(\Omega)} \right) \tag{5.2}$$

*for some positive constant $C$. Moreover, the kernel and the space orthogonal to the range consist of smooth functions.*

By adding the boundary value condition (1.6), we consider the closed subspace $H_b^s(\Omega, \mathbb{C}^{2k})$ of Sobolev spaces $H^s(\Omega, C^{2k})$ defined by

$$H_b^s(\Omega, \mathbb{C}^{2k}) := \left\{ u \in H^s(\Omega, \mathbb{C}^{2k}); D_0^{(k)*}(\nu)u = 0, D_1^{(k)*}(\nu) D_1^{(k)} u = 0 \quad \text{on} \quad \partial\Omega \right\},$$

$s > \frac{3}{2}$. The boundary value conditions above are well defined for $s > \frac{3}{2}$ by the Trace Theorem.

We know that the associated Laplacian $\square_1^{(k)}$ in (4.8) is an elliptic operator. From sections 3 and 4, we already know that boundary value problem (1.6) is regular. So we can apply Theorem 5.1 to obtain a Fredholm operator

$$T : H^{2+s}(\Omega, \mathbb{C}^{2k}) \longrightarrow H^s\left(\Omega, \mathbb{C}^{2k}\right) \oplus H^{s+\frac{3}{2}}\left(\partial\Omega, \mathbb{C}^{k+1}\right) \oplus H^{s+\frac{1}{2}}\left(\partial\Omega, \mathbb{C}^{k-1}\right) \tag{5.3}$$

defined by

$$Tu = \left( \square_1^{(k)} u, \, D_0^{(k)*}(\nu)u\Big|_{\partial\Omega}, \, D_1^{(k)*}(\nu) D_1^{(k)} u\Big|_{\partial\Omega} \right). \tag{5.4}$$

When restricted to the closed subspace $H_b^{2+s}(\Omega, \mathbb{C}^{2k}) \subset H^{2+s}(\Omega, \mathbb{C}^{2k})$, the operator $T$ gets the form $Tu = \left(\square_1^{(k)} u, 0, 0\right)$ for $u \in H_b^{2+s}(\Omega, \mathbb{C}^{2k})$. Let us prove the restriction of $T$ also Fredholm.

**Corollary 5.1.** *The operator*

$$\Box_1^{(k)} : H_b^{2+s}(\Omega, \mathbb{C}^{2k}) \longrightarrow H^s(\Omega, \mathbb{C}^{2k}) \tag{5.5}$$

*is Fredholm.*

*Proof.* Suppose that $\Box_1^{(k)}$ in (5.5) is not Fredholm. Identifying $H^s(\Omega, \mathbb{C}^{2k})$ with the subspace $\{(f, 0, 0); f \in H^s(\Omega, \mathbb{C}^{2k})\}$ of

$$\mathscr{W}_s = H^s\left(\Omega, \mathbb{C}^{2k}\right) \oplus H^{s+\frac{3}{2}}\left(\partial\Omega, \mathbb{C}^{k+1}\right) \oplus H^{s+\frac{1}{2}}\left(\partial\Omega, \mathbb{C}^{k-1}\right),$$

we see that the kernel of $\Box_1^{(k)}$ is contained in the kernel of the operator $T$ in (5.3)-(5.4), and so its dimension must be finite. Thus the cokernel of $\Box_1^{(k)}$ should be infinite dimensional.

Let us denote by $M_0$ the subspace of the Hilbert space $H^s(\Omega, \mathbb{C}^{2k})$ orthogonal to the range of $\Box_1^{(k)}$, and denote by $M$ the subspace of the Hilbert space $\mathscr{W}_s$ orthogonal to the range of $T$. Note that $H^s(\Omega, \mathbb{C}^{2k})$ is a closed subspace of the Hilbert space $\mathscr{W}_s$ by the above identification, and the range of $T$ in $\mathscr{W}_s$ is closed because it is Fredholm. So as the intersection of $H^s(\Omega, \mathbb{C}^{2k})$ and the range of $T$, the range of $\Box_1^{(k)}$ is also closed. The space $M$ is of finite dimension. Let $\{v_1, \dots v_m\}$ be a basis of $M$. Vectors $v_1, \dots v_m$ define linear functionals on $\mathscr{W}_s$, in particular on $M_0$, by the inner product of $\mathscr{W}_s$. Because $M_0$ is infinite dimensional, there must be some nonzero vector $v \in M_0$ in the kernel of these functionals, i.e., orthogonal to $M$. Consequently, $(v, 0, 0)$ belongs to the range of $T$. Namely, there exists $u \in H^{2+s}(\Omega, \mathbb{C}^{2k})$ such that $Tu = (v, 0, 0)$. This also implies that $u \in H_b^{2+s}(\Omega, \mathbb{C}^{2k})$ and $\Box_1^{(k)} u = v$, i.e., $v$ is in the range of $\Box_1^{(k)}$. This is contradict to $v \in M_0$. Thus $\Box_1^{(k)}$ has finite dimensional cokernel. The result follows. $\Box$

5.3. **Proofs of main theorems.** *Proof of Theorem 1.2.* It is sufficient to prove the theorem for $s = 0$. By Corollary 5.1, the map $\Box_1^{(k)} : H_b^2(\Omega, \mathbb{C}^{2k}) \longrightarrow L^2(\Omega, \mathbb{C}^{2k})$ is Fredholm. So its kernel, denoted by $\mathscr{K}$, is finite dimensional. Denote by $\mathscr{K}^\perp$ the orthogonal complement to $\mathscr{K}$ in $H_b^2(\Omega, \mathbb{C}^{2k})$ under the inner product of $H_b^2(\Omega, \mathbb{C}^{2k})$. Denote by $\mathscr{R}$ the range of $\Box_1^{(k)}$ in $L^2(\Omega, \mathbb{C}^{2k})$. It is a closed subspace since the cokernel of $\Box_1^{(k)}$ is also finite dimensional. Then $\Box_1^{(k)} : \mathscr{K}^\perp \to \mathscr{R}$ is bijective, and so there exists a inverse linear operator $\widetilde{N}_1^{(k)} : \mathscr{R} \to \mathscr{K}^\perp$. As a Fredholm operator, $\Box_1^{(k)} : H_b^2(\Omega, \mathbb{C}^{2k}) \longrightarrow \mathscr{R}$ is bounded, and so is its inverse $\widetilde{N}_1^{(k)}$ by the inverse operator theorem. Moreover, $\widetilde{N}_1^{(k)}$ can be extended to a bounded operator

$$N_1^{(k)} : L^2(\Omega, \mathbb{C}^{2k}) \longrightarrow \mathscr{K}^\perp \subset H_b^2(\Omega, \mathbb{C}^{2k}) \tag{5.6}$$

by setting $N_1^{(k)}$ vanishing on $\mathscr{R}^\perp$, the space orthogonal to $\mathscr{R}$ in $L^2(\Omega, \mathbb{C}^{2k})$ under the $L^2$ inner product. Namely,

$$N_1^{(k)} f = \begin{cases} \widetilde{N}_1^{(k)} f, & \text{if} \quad f \in \mathscr{R}, \\ 0, & \text{if} \quad f \in \mathscr{R}^\perp. \end{cases}$$

And there exists a positive constant $C$ such that

$$\|N_1^{(k)} f\|_{H^2(\Omega, \mathbb{C}^{2k})} \le C \|f\|_{L^2(\Omega, \mathbb{C}^{2k})} \tag{5.7}$$

for any $f \in L^2(\Omega, \mathbb{C}^{2k})$.

Now we can establish the Hodge-type orthonormal decomposition along the line in chapter 5 §9 in [16] for De Rham complex. By using the identity (2.21) in Corollary 2.21 twice, we see that if $\varphi, \varphi' \in H_b^2(\Omega, \mathbb{C}^{2k})$,

$$
\begin{aligned}
\left(\Box_1^{(k)}\varphi, \varphi'\right) &= \left(\left(D_0^{(k)}D_0^{(k)*} + D_1^{(k)*}D_1^{(k)}\right)\varphi, \varphi'\right) \\
&= \left(D_0^{(k)*}\varphi, D_0^{(k)*}\varphi'\right) + \left(D_1^{(k)}\varphi, D_1^{(k)}\varphi'\right) \qquad (5.8)\\
&= \left(\varphi, \left(D_0^{(k)}D_0^{(k)*} + D_1^{(k)*}D_1^{(k)}\right)\varphi'\right) = \left(\varphi, \Box_1^{(k)}\varphi'\right),
\end{aligned}
$$

since $D_0^{(k)*}(\nu)\varphi'|_{\partial\Omega} = D_1^{(k)*}(\nu)D_1^{(k)}\varphi|_{\partial\Omega} = 0$ and $D_0^{(k)*}(\nu)\varphi|_{\partial\Omega} = D_1^{(k)*}(\nu)D_1^{(k)}\varphi'|_{\partial\Omega} = 0$.

We will see that $\widetilde{N}_1^{(k)}$ is a self adjoint operator on $\mathscr{R}$. For any $u, v \in \mathscr{R}$, we can write $u = \Box_1^{(k)}\varphi, v = \Box_1^{(k)}\varphi' \in \mathscr{R}$ for some $\varphi, \varphi' \in H_b^2(\Omega, \mathbb{C}^{2k})$. Then by using (5.8),

$$\left(\widetilde{N}_1^{(k)}u, v\right) = \left(\widetilde{N}_1^{(k)}\Box_1^{(k)}\varphi, \Box_1^{(k)}\varphi'\right) = \left(\varphi, \Box_1^{(k)}\varphi'\right) = \left(\Box_1^{(k)}\varphi, \varphi'\right) = \left(u, \widetilde{N}_1^{(k)}v\right).$$

Consequently, $N_1^{(k)}$, as a trivial extension of $\widetilde{N}_1^{(k)}$, is also a self adjoint operator on $L^2(\Omega, \mathbb{C}^{2k})$. Because of the estimate (5.7), $N_1^{(k)}$ is compact on $L^2(\Omega, \mathbb{C}^{2k})$ by Rellich's theorem. Hence there is an orthonormal basis $\{u_j\}_{j=1}^{\infty}$ of $\mathscr{R} \subset L^2(\Omega, \mathbb{C}^{2k})$ consisting of eigenfunctions of $N_1^{(k)}$:

$$N_1^{(k)}u_j = \lambda_j u_j, \qquad \lambda_j \searrow 0.$$

Here $\lambda_j \neq 0$ since $N_1^{(k)}$ is the inverse of $\Box_1^{(k)} : \mathscr{K}^{\perp} \to \mathscr{R}$. In the view of (5.6),

$$u_j \in H_b^2(\Omega, \mathbb{C}^{2k}) \qquad \text{for} \quad \text{each} \quad j. \tag{5.9}$$

Obviously,

$$\Box_1^{(k)}u_j = \frac{1}{\lambda_j}u_j$$

Then any element of $\mathscr{K}^{\perp}$ can be written as $\sum_{j=1}^{\infty}\lambda_j a_j u_j$ for some $a_j$'s with $\sum_{j=1}^{\infty}|a_j|^2 < \infty$. Denote by $u_l^0 \in H_b^2(\Omega, \mathbb{C}^{2k})$, $l = 1, \ldots, \dim\mathscr{K}$, a basis of $\mathscr{K}$. Then $\{u_j\} \cup \{u_l^0\}$ is a basis of $H_b^2(\Omega, \mathbb{C}^{2k})$. Because $C_0^{\infty}(\Omega, \mathbb{C}^{2k}) \subset H_b^2(\Omega, \mathbb{C}^{2k})$ and $C_0^{\infty}(\Omega, \mathbb{C}^{2k})$ is dense in $L^2(\Omega, \mathbb{C}^{2k})$, we see that $H_b^2(\Omega, \mathbb{C}^{2k})$ is dense in $L^2(\Omega, \mathbb{C}^{2k})$. So $\{u_j\}\cup\{u_l^0\}$ is also a basis of $L^2(\Omega, \mathbb{C}^{2k})$. Consequently,

$$L^2(\Omega, \mathbb{C}^{2k}) = \mathscr{K} \oplus \mathscr{R}. \tag{5.10}$$

If $\psi \in \mathscr{K}$, then

$$0 = \left(\left(D_0^{(k)}D_0^{(k)*} + D_1^{(k)*}D_1^{(k)}\right)\psi, \psi\right) = \left(D_0^{(k)*}\psi, D_0^{(k)*}\psi\right) + \left(D_1^{(k)}\psi, D_1^{(k)}\psi\right)$$

by using the identity (2.21) in Corollary 2.21 since $\psi \in H_b^2(\Omega, \mathbb{C}^{2k})$. Thus $D_0^{(k)*}\psi = 0, D_1^{(k)}\psi = 0$. Note that since a function in $\mathscr{K}$ is a $C^{\infty}$ function on $\Omega$ by applying the elliptic estimate (5.2), we conclude that

$$\mathscr{K} = \mathscr{H}_{(k)}^1(\Omega). \tag{5.11}$$

By the construction of the solution operator $N_1^{(k)}$ above and the decomposition (5.10), any $\psi \in H^s(\Omega, \mathbb{C}^{2k})$ has the Hodge-type decomposition:

$$\psi = \Box_1^{(k)}N_1^{(k)}\psi + P\psi = D_0^{(k)}D_0^{(k)*}N_1^{(k)}\psi + D_1^{(k)*}D_1^{(k)}N_1^{(k)}\psi + P\psi, \tag{5.12}$$

where $P$ is the orthonomal projection to $\mathscr{K} = \mathscr{H}_{(k)}^1(\Omega)$ with respect to the $L^2$ inner product.

It is sufficient to prove orthogonality of first two terms in (5.12) for smooth functions, since $C^\infty(\overline{\Omega}, \mathbb{C}^{2k})$ is dense in $L^2(\Omega, \mathbb{C}^{2k})$ and operators $D_0^{(k)} D_0^{(k)*} N_1^{(k)}$ and $D_1^{(k)*} D_1^{(k)} N_1^{(k)}$ are both bounded in $L^2(\Omega, \mathbb{C}^{2k})$. The orthogonality follows from

$$\left(D_0^{(k)} D_0^{(k)*} N_1^{(k)} \psi, D_1^{(k)*} D_1^{(k)} N_1^{(k)} \psi\right) = \left(D_1^{(k)} D_0^{(k)} D_0^{(k)*} N_1^{(k)} \psi, D_1^{(k)} N_1^{(k)} \psi\right) = 0$$

by using the identity (2.21) in Corollary 2.21 ($D_1^{(k)*}(\nu) D_1^{(k)} N_1^{(k)} \psi|_{\partial\Omega} = 0$) for $u = D_0^{(k)} D_0^{(k)*} N_1^{(k)} \psi \in H^1(\Omega, \mathbb{C}^{2k})$ and $v = D_1^{(k)} N_1^{(k)} \psi \in H^2(\Omega, \mathbb{C}^{2k})$ when $\psi \in H^1(\Omega, \mathbb{C}^{2k})$, and using $D_1^{(k)} D_0^{(k)} = 0$. The theorem is proved. □

*Proof of Theorem 1.1.* We claim that if $D_1^{(k)} \psi = 0$ and $\psi$ is orthogonal to $\mathscr{H}^1_{(k)}(\Omega)$, then

$$\phi = D_0^{(k)*} N_1^{(k)} \psi \tag{5.13}$$

satisfies $D_0^{(k)} \phi = \psi$. Under the condition $D_1^{(k)} \psi = 0$, the second term in the decomposition (1.8) vanishes. This is because

$$\begin{aligned}\left\|D_1^{(k)*} D_1^{(k)} N_1^{(k)} \psi\right\|_{L^2}^2 &= \left(D_1^{(k)*} D_1^{(k)} N_1^{(k)} \psi, D_1^{(k)*} D_1^{(k)} N_1^{(k)} \psi\right) \\ &= \left(\psi, D_1^{(k)*} D_1^{(k)} N_1^{(k)} \psi\right) = \left(D_1^{(k)} \psi, D_1^{(k)} N_1^{(k)} \psi\right) = 0\end{aligned}$$

by using the identity (2.21) in Corollary 2.21. Here $\psi$, $D_1^{(k)} N_1^{(k)} \psi \in H^s(\Omega, \mathbb{C}^{2k})$ $(s \geq 1)$, and $N_1^{(k)} \psi \in H_b^{2+s}(\Omega, \mathbb{C}^{2k})$ implies that $D_1^{(k)*}(\nu) D_1^{(k)} N_1^{(k)} \psi|_{\partial\Omega} = 0$. The second identity comes from the orthogonality in the Hodge-type decomposition (5.12). The claim follows by $P\psi = 0$.

The estimate (1.4) follows from the estimate for the solution operator $N_1^{(k)}$ in Theorem 1.2.

Conversely, if $\psi = D_0^{(k)} \phi$ for some $\phi \in H^{s+1}(\Omega, \mathbb{C}^{k+1})$. Then $\psi \perp \mathscr{H}^1_{(k)}(\Omega)$. This is because for any $u \in \mathscr{H}^1_{(k)}(\Omega)$,

$$(\psi, u) = \left(D_0^{(k)} \phi, u\right) = \left(\phi, D_0^{(k)*} u\right) = 0$$

by using the identity (2.21) in Corollary 2.21 since $D_0^{(k)*}(\nu) u = 0$ on the boundary and $u$ and $\phi$ are both from $H^1(\Omega, \mathbb{C}^{k+1})$. □

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