

POINCARÉ'S LEMMA ON THE HEISENBERG GROUP

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Abstract: It is well known that the system $\partial_x f = a, \partial_y f = b$ on \mathbb{R}^2 has a solution if and only if the closure condition $\partial_x b = \partial_y a$ holds. In this case the solution f is the work done by the force $U = (a, b)$ from the origin to the point (x, y) .

This paper deals with a similar problem, where the vector fields ∂_x, ∂_y are replaced by the Heisenberg vector fields X_1, X_2 . In this case the sub-Riemannian system $X_1 f = a, X_2 f = b$ has a solution f if and only if the following integrability conditions hold $X_1^2 b = (X_1 X_2 + [X_1, X_2])a, X_2^2 a = (X_2 X_1 + [X_2, X_1])b$. The question addressed by this paper is whether we can provide a Poincaré-type Lemma for the Heisenberg distribution. The positive answer is given by Theorem 2, which provides a result similar to the Poincaré's Lemma in the integral form. The solution f in this case is the work done by the force vector field $aX_1 + bX_2$ along any horizontal curve from the origin to the current point.

Key words: Heisenberg distribution, work, Poincaré lemma

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1. INTRODUCTION

There are several equivalent versions of Poincaré's lemma. Some are stated in terms of exactness and closeness of 1-forms, others use conservative and gradient vector fields. The differential formulation can be found for instance in Spivak [8], while an elementary vectorial construction can be found in Yap [10]. For the sake of completeness we present next a brief review regarding a couple of elementary versions of Poincaré's lemma in dimensions 2 and 3. These will be useful in understanding of what we are trying to do in the Heisenberg case. We recall that in two dimensions the result takes the following well-known form:

Let U be an open and contractible set in \mathbb{R}^2 . Then, for a pair of smooth functions a and b

$$\partial_x b = \partial_y a \iff \begin{cases} \exists \text{ a smooth function } f \\ \text{such that } \partial_x f = a \text{ and } \partial_y f = b. \end{cases}$$

The integrability condition on the left states that the vector field $V = (a, b)$ is irrotational, while the condition on the right represents the fact that V is a gradient vector field, i.e., $V = \nabla f$, or, equivalently, V is a conservative vector field.

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The equivalent formulation using 1-forms states that the 1-form $\omega = a(x, y)dx + b(x, y)dy$ is exact, i.e., there is a smooth function f such that $\omega = df$, if and only if ω is closed, i.e., $d\omega = 0$, a condition equivalent to $\partial_x b = \partial_y a$. It is worth noting that the solution f can be constructed explicitly and has a remarkable physical significance. Let $r(t) = t(x, y) = (tx, ty) = (x(t), y(t))$, $0 \leq t \leq 1$, be the straight line segment from the origin to the point (x, y) . Consider f to be the work done by the 1-form $\omega = adx + bdy$ from the origin to the point (x, y) along the curve $r(t)$

$$f(x, y) = \int_{r(t)} \omega = \int_0^1 (a, b) \cdot \dot{r}(t) dt = \int_0^1 a(tx, ty)x + b(tx, ty)y dt.$$

An elementary computation provides

$$(1.1) \quad \partial_x f(x, y) = a(x, y) + \int_0^1 ty \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y} \right) dt$$

$$(1.2) \quad \partial_y f(x, y) = b(x, y) + \int_0^1 tx \left(\frac{\partial a}{\partial y} - \frac{\partial b}{\partial x} \right) dt.$$

Consequently, if

$$\frac{\partial a}{\partial y} = \frac{\partial b}{\partial x},$$

then $\partial_x f = a$, $\partial_y f = b$. The converse also holds true and follows as a direct consequence of the symmetry of the second derivative.

It is worthy to rewrite equations (1.1)–(1.2) in the vector format

$$(1.3) \quad \nabla f(x, y) = (a, b) + \int_0^1 \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y} \right) (ty, -tx) dt.$$

The 3-dimensional vector version of the Poincaré lemma takes the following form. Let V be a smooth vector field on \mathbb{R}^3 . Then

$$(1.4) \quad V(\mathbf{r}) = \nabla \left(\int_0^1 V(t\mathbf{r}) \cdot \mathbf{r} dt \right) + \int_0^1 \text{curl } V(t\mathbf{r}) \times t\mathbf{r} dt,$$

where $\mathbf{r} = (x, y, z)$ is the notation for points in \mathbb{R}^3 . If we denote by

$$(1.5) \quad f(\mathbf{r}) = \int_0^1 V(t\mathbf{r}) \cdot \mathbf{r} dt$$

the work done by a force expressed by the vector field V along the curve $\{t\mathbf{r}\}$, $0 \leq t \leq 1$, then (1.4) takes the equivalent form

$$(1.6) \quad \nabla f(\mathbf{r}) = V(\mathbf{r}) + \int_0^1 t\mathbf{r} \times \text{curl } V(t\mathbf{r}) dt,$$

Consequently, if $\text{curl } V = 0$, or, equivalently, if

$$\frac{\partial V^2}{\partial x_1} - \frac{\partial V^1}{\partial x_2} = 0, \quad \frac{\partial V^3}{\partial x_1} - \frac{\partial V^1}{\partial x_3} = 0, \quad \frac{\partial V^2}{\partial x_3} - \frac{\partial V^3}{\partial x_2} = 0,$$

then the work given by (1.5) satisfies

$$\nabla f(\mathbf{r}) = V(\mathbf{r}).$$

The present paper provides a variant of Poincaré's lemma in the variant of formulas (1.1)–(1.2) for the case of the Heisenberg vector fields. This is a continuation of work done in articles [2] and [3]. The point of this paper

is to advance an explicit integral formula for the Heisenberg case, which is not easy to obtain in general on sub-Riemannian manifolds.

Why would a sub-Riemannian version of Poincaré's lemma be worthy to investigate? There are several good reasons for doing this.

First, it provides a necessary and sufficient condition for a sub-elliptic system of equations to have solutions. The sub-elliptic systems are systems of linear differential equations with the number of equations smaller than the dimension of the space; section 2 introduces these systems given by (2.1). The case when the number of equations equals the dimension of the space defines an elliptic system and the existence of its solutions is provided by the classical Poincaré lemma. It is our hope that a Poincaré lemma can become in the sub-Riemannian context as powerful and influential as its elliptical, classical, version.

Second, it states the relation with the non-holonomic mechanics, since the solution of a sub-elliptical system is the work done along a horizontal curve, i.e. a curve tangent to a given distribution. For instance, the work done by moving a ball between two given positions is the work along a horizontal curve in a certain distribution of a sub-Riemannian manifold and might be related to the solution of the associated sub-elliptical system.

The plan of the paper follows. Section 2 deals with a bird's eye overview of the sub-Riemannian version of the problem and sub-Riemannian systems of equations, reviewing the current literature. Equivalent formulations using horizontal 1-forms and horizontal vector fields are presented in section 3. Section 4 contains the main result, which is a Poincaré-type formula on the Heisenberg group.

2. SUB-RIEMANNIAN SYSTEMS OF EQUATIONS

Consider a set of smooth vector fields $\{X_1, X_2, \dots, X_m\}$ on \mathbb{R}^n , where $m < n$. The vector fields span at each point $p \in \mathbb{R}^n$ a plane \mathcal{H}_p . The smooth assignment $p \rightarrow \mathcal{H}_p$ is called the *horizontal distribution* induced by $\{X_j\}$.

Consider a metric g defined on the horizontal distribution \mathcal{H} , $g : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{F}(M)$, with respect to which the vector fields $\{X_j\}$ are orthonormal, i.e. $g(X_i, X_j) = \delta_{ij}$. We note that the extension of g to \mathbb{R}^n is not unique, but this does not affect the study of sub-Riemannian geometry. The metric g can be used to measure lengths of horizontal vectors and horizontal curves, and it is called the *sub-Riemannian metric*. Here the set of smooth functions on \mathbb{R}^n is denoted by $\mathcal{F}(\mathbb{R}^n)$.

The pair (\mathcal{H}, g) provides a sub-Riemannian structure on the space \mathbb{R}^n . We shall refer to the vector fields contained in the distribution \mathcal{H} as *horizontal vector fields*. The set of all horizontal vector fields is denoted by $\Gamma(\mathcal{H})$.

Since the vector fields X_j can be viewed as the dual of m 1-forms $\{\omega_1, \dots, \omega_m\}$ on \mathbb{R}^n , one can deal with this geometry equivalently from the point of view of constraints. This approach was initiated by Vranceanu [9] in 1926, under the name of non-holonomic geometry.

A smooth curve in \mathbb{R}^n is tangent to the distribution \mathcal{H} if its velocity belongs to the plane of the distribution at each point along the curve. A curve

of this type is called a *horizontal curve*. The thermodynamic interpretation of horizontal curves is that of the adiabatic processes; they are curves along which the exchange of heat is zero, see [1], p. 74. Adiabatic processes are important components of Carnot's proposed engine in 1820s. Carathéodory provided in his 1919 paper a necessary condition for the Pfaff system associated with a horizontal distribution to admit connectivity by horizontal curves. This is the reason why sub-Riemannian geometry is also known under the name of Carnot-Carathéodory geometry.

The following condition on \mathcal{H} was used independently by Chow and Rashevskii ([5], [7]) to prove the global connectedness of \mathbb{R}^n by horizontal curves. The same condition was also used by Hörmander [6] as a sufficient, but not necessary condition for the differential operator $\sum_j X_j^2$ to be hypoelliptic.

The horizontal distribution \mathcal{H} satisfies the *bracket generating condition* if the vector fields X_j , together with finitely many of their iterated brackets span the tangent space of \mathbb{R}^n at each point. This means that for each $p \in \mathbb{R}^n$, there is an $r > 1$ such that

$$X_i, \dots, [X_i, X_j], \dots, [X_i, [X_j, X_k]], \dots, [\dots [X_{i_1}, \dots, [X_{i_r}, X_{i_{r+1}}]] \dots]$$

span $T_p \mathbb{R}^n$. If $r = r(p)$ is the smallest integer with this property at the point p , then the number $k(p) = r(p) + 1$ is called the *step* of the distribution at the point p . Some authors use also the equivalent term of order of nonholonomy. It is worth noting that $k(p)$ depends on the point p and might exhibit discontinuities from point to point.

Recall now the open problem asked in [1], p. 51:

Given m smooth functions $a_j \in \mathcal{F}(\mathbb{R}^n)$, find a function $f \in \mathcal{F}(\mathbb{R}^n)$ that satisfies the system

$$(2.1) \quad \begin{aligned} X_1(f) &= a_1 \\ &\dots\dots\dots \\ X_m(f) &= a_m. \end{aligned}$$

Since the number of equations is smaller than the space dimension, $m < n$, the previous system is called sub-elliptic.

If \mathcal{H} satisfies the *bracket generating condition*, then the solution f , if exists, is unique up to an additive constant. The regularity of the solution follows from Hörmander's theorem (see [6]) that yields the hypoellipticity of the operator $\sum_j X_j^2$.

The solution existence is a question of a delicate matter, and was solved only in a few particular cases. The existence of solutions is equivalent to a set of integrability conditions. The integrability conditions for the case of the Heisenberg group can be found in [1], p. 53 (see Theorem 2.9.8):

Theorem 1. *Let $X_1 = \partial_x - 2y\partial_t$, $X_2 = \partial_y + 2x\partial_t$ be the Heisenberg vector fields on \mathbb{R}^3 . The system $X_1 f = a$, $X_2 f = b$ has a solution $f \in \mathcal{F}(\mathbb{R}^3)$ if and only if*

$$\begin{aligned} X_1^2 b &= (X_1 X_2 + [X_1, X_2])a \\ X_2^2 a &= (X_2 X_1 + [X_2, X_1])b. \end{aligned}$$

The paper [2] uses complexes of differential operators and symmetry reductions from the Heisenberg and Engel distributions to provide integrability conditions for the Grushin and Martinet distributions.

A direct method to prove the integrability conditions for Grushin distributions of step 2 and 3 as well as for Heisenberg-type distributions on 3-dimensional manifolds is presented in the paper [3].

From the analysis performed in the aforementioned papers we know that in the case of distributions where the step exhibits jumps the exactness conditions are non-classical, in the sense that there are some extra functions that appear in the integrability relations. This occurs in the case of Grushin and Martinet distributions and this behavior is expected for all distributions with nonconstant step. On the other side, for constant step distributions (e.g. Heisenberg, Engel, etc.) we expect classical integrability conditions that can be expressed only in terms of vector fields $\{X_j\}$.

3. EQUIVALENT FORMULATIONS

This section reviews a few operators used in sub-Riemannian geometry. We shall apply them to provide equivalent formulations for the solution existence problem that was presented in section 2.

Let $f \in \mathcal{F}(\mathbb{R}^n)$. The following definition provides two equivalent ways to measure the change of the function f along a horizontal distribution.

Definition 1. *The horizontal gradient of f is the horizontal vector field $\nabla_h f$ defined by*

$$g(\nabla_h f, X) = X(f), \quad \forall X \in \Gamma(\mathcal{H}).$$

The horizontal differential of f is a horizontal 1-form (a form acting on horizontal vector fields) denoted by $d_h f$ and defined by

$$d_h f(X) = X(f), \quad \forall X \in \Gamma(\mathcal{H}).$$

We note that $g(\nabla_h f, X) = d_h f(X)$, for all $X \in \Gamma(\mathcal{H})$. The previous operators can be represented intrinsically as follows.

Proposition 1. *Let $\{\omega_j\}$ be dual 1-forms on \mathbb{R}^n associated with the basic vector fields $\{X_j\}$, i.e. $\omega_j(X_i) = \delta_j^i$. We have*

$$(i) \quad \nabla_h f = \sum_j X_j(f) X_j$$

$$(ii) \quad d_h f = \sum_j X_j(f) \omega_j.$$

Proof. (i) Writing with respect to the basis $\{X_1, \dots, X_m\}$ yields

$$\nabla_h f = \sum_j g(\nabla_h f, X_j) X_j = \sum_j X_j(f) X_j.$$

(ii) It suffices to check the formula on a basis of \mathcal{H}_p at a point p . Applying both sides to X_i , the left side becomes

$$d_h f(X_i) = X_i(f)$$

while the right side yields the same formula

$$\sum_j X_j(f) \omega_j(X_i) = \sum_j X_j(f) \delta_i^j = X_i(f).$$

□

The next definition states a dual terminology.

Definition 2. A horizontal vector field $U \in \Gamma(\mathcal{H})$ is called *conservative*, if there is a function $f \in \mathcal{F}(M)$ such that $U = \nabla_h f$.
A horizontal 1-form ω is called *exact* if there is a function $f \in \mathcal{F}(M)$ such that $\omega = d_h f$.

It is worth noting that there is a bijective correspondence between conservative horizontal vector fields U and exact horizontal 1-forms ω ; this correspondence is implied by the formula

$$g(U, X) = \omega(X), \quad \forall X \in \Gamma(\mathcal{H}).$$

The system (2.1) can be written equivalently as $\nabla_h f = U$, with $U = \sum_j a_j X_j$. Therefore, the fact that U is conservative is equivalent to the existence of solutions for the system (2.1).

On the other hand, the existence of solutions for the system (2.1) is equivalent to the exactness of a horizontal 1-form: *given a horizontal 1-form $\eta = \sum_j a_j(x)\omega_j$, find $f \in \mathcal{F}(\mathbb{R}^n)$ such that $d_h f = \eta$.*

4. MAIN RESULT

The goal of this section is to obtain a result on the Heisenberg group that is an analog of the vectorial formula (1.6), and which will be given by Theorem 2. The Heisenberg distribution \mathcal{H} is generated on \mathbb{R}^3 by the vector fields

$$X_1 = \partial_x - 2y\partial_z, \quad X_2 = \partial_y + 2x\partial_z.$$

Since $X_1, X_2, [X_1, X_2] = 4\partial_z$ are linearly independent at every point, the distribution \mathcal{H} is of constant step equal to 2 at each point. The dual coframe of $X_1, X_2, X_3 = \partial_z$ is given by

$$\omega^1 = dx, \quad \omega^2 = dy, \quad \omega^3 = 2ydx - 2xdy + dz.$$

The differential of a function f can be written in terms of this coframe as

$$\begin{aligned} df &= X_1(f)\omega^1 + X_2(f)\omega^2 + X_3(f)\omega^3 \\ &= df(X_1)\omega^1 + df(X_2)\omega^2 + df(X_3)\omega^3. \end{aligned}$$

Since the horizontal distribution is given by $\mathcal{H} = \ker \omega^3$, the constraint $\omega^3 = 0$ implies

$$d_h f = X_1(f)\omega^1 + X_2(f)\omega^2.$$

The dual object to $d_h f$ is the horizontal gradient of f , which is given by $\nabla_h f = X_1(f)X_1 + X_2(f)X_2$.

A curve $c(t) = (x(t), y(t), z(t))$ is horizontal if the non-holonomic constraint $\omega^3(\dot{c}) = 0$, i.e., $2y(t)\dot{x}(t) - 2x(t)\dot{y}(t) + \dot{z}(t) = 0$ holds. In this case the velocity of the curve can be written as $\dot{c}(t) = \dot{x}(t)X_1 + \dot{y}(t)X_2$. The existence of smooth horizontal curves joining the origin with any given point (x, y, z) is proved, for instance, in [4] p. 10.

It is useful to note that if $U = U^1X_1 + U^2X_2$ and $V = V^1X_1 + V^2X_2$ are two horizontal vector fields and g denotes the sub-Riemannian metric, then $g(U, V) = U^1V^1 + U^2V^2$.

Since $[X_1, X_2] = 4\partial_z$, the distribution is step 2 everywhere. The system

$$\begin{aligned} X_1 f &= a \\ X_2 f &= b \end{aligned}$$

can be written, after adding the additional condition $[X_1, X_2]f = c$, as

$$(4.1) \quad \begin{aligned} \partial_x f - 2y\partial_z f &= a \\ \partial_y f + 2x\partial_z f &= b \\ 4\partial_z f &= c. \end{aligned}$$

Setting

$$\begin{aligned} a_1 &= a + 2y\partial_z f = a + yc/2 \\ b_1 &= b - 2x\partial_z f = b - xc/2 \\ c_1 &= c/4 = (X_1 b - X_2 a)/4 \end{aligned}$$

the system (4.1) becomes

$$\begin{aligned} \partial_x f &= a_1 \\ \partial_y f &= b_1 \\ \partial_z f &= c_1. \end{aligned}$$

Consider the vector field $V = (a_1, b_1, c_1)$ on \mathbb{R}^3 . We note that the equation $\nabla f = V$ has solutions if and only if $\text{curl } V = 0$.

Theorem 2. *Let $X_1 = \partial_x - 2y\partial_z$, $X_2 = \partial_y + 2x\partial_z$ be the Heisenberg vector fields on \mathbb{R}^3 and consider*

$$(4.2) \quad f(\mathbf{r}) = \int_0^1 V(t\mathbf{r}) \cdot \mathbf{r} dt.$$

Then

$$\begin{aligned} (X_1 f)(\mathbf{r}) &= a(\mathbf{r}) + \int_0^1 \frac{tz}{4} \left(X_1^2 b - (X_1 X_2 + [X_1, X_2])a \right)(t\mathbf{r}) dt \\ (X_2 f)(\mathbf{r}) &= b(\mathbf{r}) - \int_0^1 \frac{tz}{4} \left(X_2^2 a - (X_2 X_1 + [X_2, X_1])b \right)(t\mathbf{r}) dt. \end{aligned}$$

If the integrability conditions

$$\begin{aligned} X_1^2 b &= (X_1 X_2 + [X_1, X_2])a \\ X_2^2 a &= (X_2 X_1 + [X_2, X_1])b \end{aligned}$$

hold, then

$$X_1 f = a, \quad X_2 f = b$$

with

$$f(x, y, z) = \int_0^1 g(U_{\gamma(t)}, \dot{\gamma}(t)) dt,$$

where $g(\cdot, \cdot)$ denotes the sub-Riemannian metric and $U = aX_1 + bX_2$.

Proof. We start by expressing the integrability conditions in terms of local coordinates. A straightforward computation provides

$$(4.3) \quad \begin{aligned} X_1^2 b - (X_1 X_2 + [X_1, X_2])a &= (-6\partial_z + 4xy\partial_z^2 + 2y\partial_y\partial_z - 2x\partial_x\partial_z - \partial_x\partial_y)a \\ &+ (4y^2\partial_z^2 - 4y\partial_x\partial_z + \partial_x^2)b, \end{aligned}$$

$$(4.4) \quad \begin{aligned} X_2^2 a - (X_2 X_1 + [X_2, X_1])b &= (4x^2\partial_z^2 + 4x\partial_y\partial_z + \partial_y^2)a \\ &+ (6\partial_z + 4xy\partial_z^2 + 2y\partial_y\partial_z - 2x\partial_x\partial_z - \partial_x\partial_y)b. \end{aligned}$$

Multiplying by the matrix $A = \begin{pmatrix} 1 & 0 & -2y \\ 0 & 1 & 2x \end{pmatrix}$ on the left of the relation

$$(4.5) \quad \nabla f(\mathbf{r}) = V(\mathbf{r}) + \int_0^1 \mathbf{tr} \times \operatorname{curl} V(\mathbf{tr}) dt,$$

we obtain

$$\begin{aligned} A\nabla f(\mathbf{r}) &= AV(\mathbf{r}) + \int_0^1 A(\mathbf{tr} \times \operatorname{curl} V(\mathbf{tr})) dt \iff \\ \begin{pmatrix} X_1 f \\ X_2 f \end{pmatrix}(\mathbf{r}) &= \begin{pmatrix} a \\ b \end{pmatrix}(\mathbf{r}) + \int_0^1 \begin{pmatrix} \langle (1, 0, -2y), \mathbf{tr} \times \operatorname{curl} V(\mathbf{tr}) \rangle \\ \langle (0, 1, 2x), \mathbf{tr} \times \operatorname{curl} V(\mathbf{tr}) \rangle \end{pmatrix} dt. \end{aligned}$$

By the circular permutation invariance property of the mixed product of vectors, i.e., $\langle u, v \times w \rangle = \langle w, u \times v \rangle$, the previous formula becomes

$$(4.6) \quad \begin{pmatrix} X_1 f \\ X_2 f \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} + \int_0^1 \begin{pmatrix} \langle \operatorname{curl} V(\mathbf{tr}), (1, 0, -2y) \times \mathbf{tr} \rangle \\ \langle \operatorname{curl} V(\mathbf{tr}), (0, 1, 2x) \times \mathbf{tr} \rangle \end{pmatrix} dt.$$

The explicit computation of the components of $\operatorname{curl} V = ((\operatorname{curl} V)^1, (\operatorname{curl} V)^2, (\operatorname{curl} V)^3)$ is given in the following and will be used shortly

$$\begin{aligned} (\operatorname{curl} V)^1 &= \partial_y c - \partial_z b \\ &= \frac{1}{4}(-4x^2\partial_z^2 - 4x\partial_y\partial_z - \partial_y^2)a \\ &+ \frac{1}{4}(-6\partial_z - 4xy\partial_z^2 - 2y\partial_y\partial_z + 2x\partial_x\partial_z + \partial_x\partial_y)b; \end{aligned}$$

$$\begin{aligned} (\operatorname{curl} V)^2 &= \partial_z a - \partial_x c \\ &= \frac{1}{4}(6\partial_z - 4xy\partial_z^2 - 2y\partial_y\partial_z + 2x\partial_x\partial_z + \partial_x\partial_y)a \\ &+ \frac{1}{4}(-4y^2\partial_z^2 + 4y\partial_x\partial_z - \partial_x^2)b; \end{aligned}$$

$$\begin{aligned} (\operatorname{curl} V)^3 &= \partial_x b - \partial_y a \\ &= \frac{1}{2}(6x\partial_z + 2xy\partial_y\partial_z + y\partial_y^2 + 2x^2\partial_x\partial_z + x\partial_x\partial_y)a \\ &+ \frac{1}{2}(6y\partial_z + 2y^2\partial_y\partial_z + 2xy\partial_x\partial_z - y\partial_x\partial_y - x\partial_x^2)b. \end{aligned}$$

Using the valuation of the mixed product as a determinant we have

$$\begin{aligned}
\langle \operatorname{curl} V(\mathbf{r}), (1, 0, -2y) \times \mathbf{r} \rangle &= \begin{vmatrix} (\operatorname{curl} V)^1 & (\operatorname{curl} V)^2 & (\operatorname{curl} V)^3 \\ 1 & 0 & -2y \\ x & y & z \end{vmatrix} \\
&= 2y^2(\operatorname{curl} V)^1 - (z + 2xy)(\operatorname{curl} V)^2 + y(\operatorname{curl} V)^3 \\
&= \frac{z}{4} \left\{ (-6\partial_z + 4xy\partial_z^2 + 2y\partial_y\partial_z - 2x\partial_x\partial_z - \partial_x\partial_y)a \right. \\
&\quad \left. + (4y^2\partial_z^2 - 4y\partial_x\partial_z + \partial_x^2)b \right\} \\
&= \frac{z}{4} (X_1^2 b - (X_1 X_2 + [X_1, X_2])a)(\mathbf{r}),
\end{aligned}$$

where we used (4.3). A similar computation applies to the second component

$$\begin{aligned}
\langle \operatorname{curl} V(\mathbf{r}), (0, 1, 2x) \times \mathbf{r} \rangle &= \begin{vmatrix} (\operatorname{curl} V)^1 & (\operatorname{curl} V)^2 & (\operatorname{curl} V)^3 \\ 0 & 1 & 2x \\ x & y & z \end{vmatrix} \\
&= (z - 2xy)(\operatorname{curl} V)^1 + 2x^2(\operatorname{curl} V)^2 - x(\operatorname{curl} V)^3 \\
&= -\frac{z}{4} \left\{ (4x^2\partial_z^2 + 4x\partial_y\partial_z + \partial_y^2)a \right. \\
&\quad \left. + (6\partial_z + 4xy\partial_z^2 + 2y\partial_y\partial_z - 2x\partial_x\partial_z - \partial_x\partial_y)b \right\} \\
&= -\frac{z}{4} (X_2^2 a - (X_2 X_1 + [X_2, X_1])b)
\end{aligned}$$

by formula (4.4). Substituting back into (4.6) yields

$$\begin{aligned}
(X_1 f)(\mathbf{r}) &= a(\mathbf{r}) + \int_0^1 \frac{tz}{4} (X_1^2 b - (X_1 X_2 + [X_1, X_2])a)(t\mathbf{r}) dt \\
(X_2 f)(\mathbf{r}) &= b(\mathbf{r}) - \int_0^1 \frac{tz}{4} (X_2^2 a - (X_2 X_1 + [X_2, X_1])b)(t\mathbf{r}) dt.
\end{aligned}$$

Let

$$(4.7) \quad f(\mathbf{r}) = \int_0^1 V(t\mathbf{r}) \cdot \mathbf{r} dt$$

be the work done by a force given by the vector field V along the curve $\{t\mathbf{r}\}$, $0 \leq t \leq 1$, and assume $\operatorname{curl} V = 0$. Then $\nabla f = V$ on \mathbb{R}^3 .

We review next a few vectorial calculus notions. Let $c : [0, 1] \rightarrow \mathbb{R}^n$ be a smooth curve and Y be a smooth vector field. The fundamental theorem of calculus states that if $Y = \nabla f$, then the integral depends only on the end point values of f

$$\int_c Y \cdot dc = \int_0^1 \langle Y, c'(t) \rangle dt = f(c(1)) - f(c(0)).$$

Assume $\operatorname{curl} V = 0$, so the system becomes $\nabla f = V$ on \mathbb{R}^3 . Choosing $c(t) = t\mathbf{r} = (tx, ty, tz)$ we have the solution given by

$$f(\mathbf{r}) = \int_0^1 V(t\mathbf{r}) \cdot \mathbf{r} dt = \int_0^1 \langle V, c'(t) \rangle dt = f(c(1)) - f(c(0)).$$

The result holds for any other curve with the same end points.

If $\gamma(t) = (x(t), y(t), z(t))$ is a horizontal curve, then $\dot{\gamma} = \dot{x}X_1 + \dot{y}X_2$ and hence $\dot{z} = 2x\dot{y} - 2\dot{x}y$. Let $U = aX_1 + bX_2$ be the horizontal vector field associated with our system. Then

$$\begin{aligned} \int_0^1 \langle V, \dot{\gamma}(t) \rangle dt &= \int_0^1 (a_1\dot{x}(t) + b_1\dot{y}(t) + c_1\dot{z}(t)) dt \\ &= \int_0^1 \left((a + \frac{1}{2}yc)\dot{x}(t) + (b - \frac{1}{2}cx)\dot{y}(t) + \frac{1}{4}c\dot{z}(t) \right) dt \\ &= \int_0^1 (a\dot{x} + b\dot{y}) dt + \frac{c}{4} \int_0^1 \underbrace{(2y\dot{x} - 2x\dot{y} + \dot{z})}_{=0} dt \\ &= \int_0^1 g(U_{\gamma(t)}, \dot{\gamma}(t)) dt, \end{aligned}$$

where $g(\cdot, \cdot)$ stands for the sub-Riemannian metric. Hence, if the integrability conditions hold, i.e., if

$$\begin{aligned} X_1^2 b &= (X_1 X_2 + [X_1, X_2])a \\ X_2^2 a &= (X_2 X_1 + [X_2, X_1])b, \end{aligned}$$

then the solution is given by the work done by the horizontal vector field $U = aX + bY$ along any horizontal curve $\gamma(t) = (x(t), y(t), z(t))$ with $c(0) = (0, 0, 0)$, $c(1) = (x, y, z)$, and this is given by

$$f(x, y, z) = \int_0^1 g(U_{\gamma(t)}, \dot{\gamma}(t)) dt.$$

This represents the work done by the force U along any horizontal curve joining the origin and (x, y, z) . \square

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