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L^q-EXTENSIONS OF L^p-SPACES BY FRACTIONAL DIFFUSION EQUATIONS

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Abstract. Based on the geometric-measure-theoretic analysis of a new L^p-type capacity defined in the upper-half Euclidean space, this note characterizes a nonnegative Randon measure μ on $\mathbb{R}^{1+n}_+$ such that the extension $R_\alpha L^p(\mathbb{R}^n) \subseteq L^q(\mathbb{R}^{1+n}_+, \mu)$ holds for $(\alpha, p, q) \in (0,1)\times(1,\infty)\times(1,\infty)$ where $u = R_\alpha f$ is the weak solution of the fractional diffusion equation $(\partial_t + (-\Delta_x)^\alpha)u(t,x) = 0$ in $\mathbb{R}^{1+n}_+$ subject to $u(0,x) = f(x)$ in $\mathbb{R}^n$.

1. Introduction.

1.1. The fractional diffusion kernel.

For a natural number n and $\alpha \in (0,1]$, suppose $\mathbb{R}^{1+n}_+ = (0,\infty)\times\mathbb{R}^n$ is the upper half space of the $1+n$ dimensional Euclidean space $\mathbb{R}^{1+n}$. As an important non-local operator modelling the local dynamics of many complicated processes in physics, engineering and applied sciences (see e.g. [5, 25]), the $(0,1] \ni \alpha$-th order Laplacian $(-\Delta_x)^\alpha$ in the space variable x is defined by

$$(-\Delta_x)^\alpha u(\cdot,x) = \mathcal{F}^{-1}(|\xi|^{2\alpha}\mathcal{F}u(\cdot,\xi))(x) \quad \forall x \in \mathbb{R}^n,$$

where $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ its inverse:

$$\begin{cases} \mathcal{F}(g)(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\cdot y} g(y)\,dy; \\ \mathcal{F}^{-1}(g)(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot y} g(y)\,dy. \end{cases}$$

The fractional diffusion kernel or the fundamental solution of the so-called fractional diffusion operator

$$L^{(\alpha)} = \partial_t + (-\Delta_x)^\alpha$$

is:

$$K^{(\alpha)}_t(x) = \begin{cases} (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot y - t|y|^{2\alpha}}\,dy & \text{for} \quad (t,x) \in \mathbb{R}^{1+n}_+; \\ \delta_x \text{ (the Dirac measure)} & \text{for} \quad (t,x) \in \{0\}\times\mathbb{R}^n. \end{cases}$$

In particular,

$$K^{(1)}_t(x) = \begin{cases} (4\pi t)^{-n/2} e^{-|x|^2/(4t)} & \text{for} \quad (t,x) \in \mathbb{R}^{1+n}_+; \\ \delta_x \text{ (the Dirac measure)} & \text{for} \quad (t,x) \in \{0\}\times\mathbb{R}^n, \end{cases}$$

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and

$$K_t^{(1/2)}(x)=\begin{cases}\pi^{-(1+n)/2}\Gamma\big((n+1)/2\big)t(t^2+|x|^2)^{-(1+n)/2} \quad \text{for} \quad (t,x)\in\mathbb{R}^{1+n}_+;\\ \delta_x \ (\text{the Dirac measure}) \quad \text{for} \quad (t,x)\in\{0\}\times\mathbb{R}^n,\end{cases}$$

are the Gauss and Poisson kernels respectively with $\Gamma(\cdot)$ being the gamma function. Though $K_t^{(\alpha)}(x)$ has no explicit formula in physical variables for α other than 1/2 and 1 (cf. [10, 12, 19, 20, 21, 22, 23, 26, 28, 33]), the following estimates (see [6, Theorem 2.1] or [11, 17, 31]) under $0<\alpha<1$:

$$\begin{cases}K_t^{(\alpha)}(x)=\frac{K_1^{(\alpha)}\big(t^{-(2\alpha)^{-1}}x\big)}{t^{\frac{n}{2\alpha}}}\approx\min\{t^{\frac{-n}{2\alpha}},t|x|^{n+2\alpha}\}\approx\frac{t}{(t^{\frac{1}{2\alpha}}+|x|)^{n+2\alpha}} \quad \forall(t,x)\in\mathbb{R}^{1+n}_+;\\ \int_{\mathbb{R}^n}K_t^{(\alpha)}(x)\,dx=1 \quad \forall \quad t\in(0,\infty),\end{cases}$$

will be important to us, and hence $0<\alpha<1$ will be assumed from now on.

1.2. **An extension by $L^{(\alpha)}$.** Denote by

$$\tilde{L}^{(\alpha)}=-\partial_t+(-\Delta_x)^\alpha$$

the dual operator of $L^{(\alpha)}$, which is determined by

$$-\partial_t\phi(t,x)+\left(\frac{(1-\alpha)2^{2\alpha}\Gamma\big(\frac{n+2\alpha}{2}\big)}{\pi^{n/2}\Gamma(1-\alpha)}\right)\lim_{\epsilon\to0}\int_{\{y\in\mathbb{R}^n:|y|>\epsilon\}}\frac{\phi(t,x+y)-\phi(t,x)}{|y|^{-n-2\alpha}}\,dy$$

acting on $\phi\in C_0^\infty(\mathbb{R}^{1+n}_+)$.

For $f\in L^p(\mathbb{R}^n)$ with $1\le p\le\infty$, we consider its extension $u:\mathbb{R}^{1+n}_+\mapsto\mathbb{R}$ via

$$\begin{cases}L^{(\alpha)}u(t,x)=0 \quad \forall(t,x)\in\mathbb{R}^{1+n}_+;\\ u(0,x)=f(x) \quad \forall x\in\mathbb{R}^n,\end{cases}$$

in the sense of (cf. [14, 20]):

$$\iint_{[0,\infty)\times\mathbb{R}^n}u(t,x)\tilde{L}^{(\alpha)}\phi(t,x)\,dxdt=-\int_{\mathbb{R}^n}f(x)\phi(0,x)\,dx \quad \forall\phi\in C_0^\infty([0,\infty)\times\mathbb{R}^n)$$

Formally, this extension can be represented as

$$u(t,x)=R_\alpha f(t,x)=e^{-t(-\Delta_x)^\alpha}f(x)=K_t^{(\alpha)}*f(x),$$

where $*$ stands for the convolution acting on the space variable.

Lemma 1.1.
(i) *If*

$$\begin{cases}1\le p\le\tilde{p}<\frac{np}{n-\min\{n,2\alpha\}};\\ \frac{1}{\tilde{q}}=\big(\frac{n}{2\alpha}\big)\big(\frac{1}{p}-\frac{1}{\tilde{p}}\big),\end{cases}$$

then

$$\|R_\alpha f\|_{L_t^{\tilde{q}}L_x^{\tilde{p}}(\mathbb{R}^{1+n}_+)}\lesssim\|f\|_{L^p(\mathbb{R}^n)}.$$

(ii) *If $n=2\alpha$ then*

$$\|f\|^2_{L^2(\mathbb{R}^n)}\gtrsim\begin{cases}\int_0^\infty\|R_\alpha f(t,\cdot)\|^2_{BMO(\mathbb{R}^n)}\,dt;\\ \int_0^\infty t^{-2/p}\|R_\alpha f(t,\cdot)\|^2_{L^p(\mathbb{R}^n)}\,dt \quad \textit{for} \quad p\in(2,\infty).\end{cases}$$

(iii) *If $f\in C_0^\infty(\mathbb{R}^n)$ and $p\in[1,\infty)$ then:*

(a) *$R_\alpha f$ is continuous in $\mathbb{R}^{1+n}_+$ and $R_\alpha f(t,\cdot)$ is infinitely smooth for $t\in\mathbb{R}_+$;*

(b) *$t\mapsto\|R_\alpha f(t,\cdot)\|_{L^p(\mathbb{R}^n)}$ is a decreasing function on $\mathbb{R}_+$;*

(c) *$\|R_\alpha f(t,\cdot)\|_{L^p(\mathbb{R}^n)}\le\|f\|_{L^p(\mathbb{R}^n)}$ holds for any $t\in\mathbb{R}_+$;*

(d) *$r\mapsto r^{-2\alpha}\int_0^{r^{2\alpha}}\|R_\alpha f(t,\cdot)\|_{L^p(\mathbb{R}^n)}\,dt$ is a decreasing function on $\mathbb{R}_+$.*

Proof. (i) This is just [19, Lemma 3.2] which extends the case $n = 2$, $\alpha \in (1/2, 1]$, and $r^{-1} = \alpha - 2^{-1}$ of [27, Lemma 3.2].

(ii) This comes from [32, Theorems 1.6-1.7].

(iii) The assertion (a) is presented in [16, Theorem 1.2 (i)] and its proof. Using the argument for [13, Lemma 2.5] we readily see

$$\frac{d}{dt}\|R_\alpha f(t,\cdot)\|^p_{L^p(\mathbb{R}^n)} = -p\int_{\mathbb{R}^n}|R_\alpha f(t,x)|^{p-2}R_\alpha f(t,x)(-\Delta_x)^\alpha R_\alpha f(t,x)\,dx \le 0,$$

whence getting (b). Of course, (c) is a consequence of (b). To verify (d), we just compute

$$r^{-2\alpha}\int_0^{r^{2\alpha}}\|R_\alpha f(t,\cdot)\|_{L^p(\mathbb{R}^n)}\,dt = \int_0^1\|R_\alpha f(r^{2\alpha}t,\cdot)\|_{L^p(\mathbb{R}^n)}\,dt$$

and use (b).

□

In the above and below, we have used the following notations.

- $C_0^\infty(\mathbb{X})$ stands for all infinitely smooth functions with compact support in a given space $\mathbb{X}$;
- $L^p(\mathbb{R}^n)$ denotes the usual Lebesgue p-space with respect to the space variable x;
- $L_t^{p_2}L_x^{p_1}(\mathbb{R}^{1+n}_+)$ is the mixed $(1 \le p_1, p_2 < \infty)$-Lebesgue space of all functions F on $\mathbb{R}^{1+n}_+$ with
$$\|F\|_{L_t^{p_2}L_x^{p_1}(\mathbb{R}^{1+n}_+)} = \left(\int_{\mathbb{R}_+}\left(\int_{\mathbb{R}^n}|F(t,x)|^{p_1}\,dx\right)^{\frac{p_2}{p_1}}dt\right)^{\frac{1}{p_2}} < \infty,$$
in which an appropriate modification is needed when p_1 or p_2 is ∞;
- $$\|g\|_{BMO(\mathbb{R}^n)} = \left(\sup_Q \ell(Q)^{-n}\int_Q|f(x) - f_Q|^2\,dx\right)^{\frac{1}{2}} < \infty,$$
where f_Q stands for the average of f over Q and the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes and $\ell(Q)$ as their sidelengths;
- $\mathsf{X} \lesssim \mathsf{Y}$ means that there is a constant $c > 0$ such that $\mathsf{X} \le c\mathsf{Y}$ - moreover, if $\mathsf{Y} \lesssim \mathsf{X}$ is also valid then one writes $\mathsf{X} \approx \mathsf{Y}$.

Partially motivated by Lemma 1.1 plus [1, 2, 4, 7, 8, 9, 16, 29, 30], we are about to get a much better understanding of the above extension $u = R_\alpha f$ of $f \in L^p(\mathbb{R}^n)$ with $(p, \alpha) \in [1, \infty] \times (0, 1)$ in the frame of function space embeddings. Our approach is rooted in the well-developed theory for the traces and imbeddings associated with the classical L^p-capacitities; see also [3]. More precisely, in Section 2, we use $R_\alpha f$ to not only introduce a new kind of L^p-capacity over $\mathbb{R}^{1+n}_+$ but also explore its basic geometric-measure-theoretic properties. And, Section 3 is devoted to treating the embedding of $R_\alpha L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^{1+n}_+, \mu)$ with μ being a nonnegative Randon measure on $\mathbb{R}^{1+n}_+$.

2. L^p-capacities in $\mathbb{R}^{1+n}_+$.

2.1. Capacitary definition and its dual form.

The following defines the so-called L^p type capacity.

Definition 2.1. For a compact subset K of $\mathbb{R}^{1+n}_+$, let

$$C_p^{(\alpha)}(K) = \inf\left\{\|f\|^p_{L^p(\mathbb{R}^n)} : \quad f \ge 0 \ \&\ R_\alpha f \ge 1_K\right\},$$

where 1_K is the characteristic function of K and $1 \le p, q < \infty$. When O is an open subset of $\mathbb{R}^n$, one defines

$$C_p^{(\alpha)}(O) = \sup\left\{C_p^{(\alpha)}(K) : \text{ compact } \ K \subset O\right\}$$

and hence for any set $E \subseteq \mathbb{R}^n$, one sets

$$C_p^{(\alpha)}(E) = \inf\left\{C_p^{(\alpha)}(O) : \text{ open } \ O \supset E\right\}.$$

To establish the adjoint formulation of the foregoing definition, we need to find out adjoint operator R_α^* of the operator R_α. Note that

$$\iint_{\mathbb{R}^{1+n}_+} R_\alpha f(t,x)g(t,x)\,dtdx = \int_{\mathbb{R}^n} f(x)\left(\iint_{\mathbb{R}^{1+n}_+} K_t^{(\alpha)}(z-x)g(t,z)\,dtdz\right)dx$$

holds for all $f\in C_0^\infty(\mathbb{R}^n)$ and $g\in C_0^\infty(\mathbb{R}^{1+n}_+)$. So, R_α^* is defined via

$$\big(R_\alpha^* g\big)(x)=\iint_{\mathbb{R}^{1+n}_+} K_t^{(\alpha)}(z-x)g(t,z)\,dtdz\quad\forall\quad g\in C_0^\infty(\mathbb{R}^{1+n}_+).$$

This definition of R_α^* can be extended to the class of Borel measures μ with compact support in $\mathbb{R}^{1+n}_+$. In fact, note that if f is continuous and has a compact support in $\mathbb{R}^n$ and $\|\mu\|_1$ stands for the total variation of μ then a simple calculation with

$$K_t^{(\alpha)}(x)\approx t(t^{1/(2\alpha)}+|x|)^{-n-2\alpha}\quad\forall\quad (t,x)\in\mathbb{R}^{1+n}_+,$$

gives

$$\left|\int_{\mathbb{R}^{1+n}_+} R_\alpha f\,d\mu\right|\lesssim\big(\sup_{x\in\mathbb{R}^n}|f(x)|\big)\|\mu\|_1.$$

Hence an application of the Riesz representation theorem yields a Borel measure ν on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^{1+n}_+} R_\alpha f\,d\mu=\int_{\mathbb{R}^n} f\,d\nu\quad\forall\quad f\geq 0.$$

This actually says that $\nu(x)=R_\alpha^*\mu$ is determined by

$$R_\alpha^*\mu(x)=\iint_{\mathbb{R}^{1+n}_+} K_t^{(\alpha)}(z-x)\,d\mu(t,z).$$

The above analysis leads to a dual characterization of the capacity.

Proposition 1. *Given $p\in(1,\infty)$ and a compact subset K of $\mathbb{R}^{1+n}_+$, let $p'=p/(p-1)$ and $\mathcal{M}_+(K)$ be the class of all nonnegative Radon measures supported by K.*
(i)

$$C_p^{(\alpha)}(K)=\sup\big\{\|\mu\|_1^p:\ \mu\in\mathcal{M}_+(K)\ \ \&\ \ \|R_\alpha^*\mu\|_{L^{p'}(\mathbb{R}^n)}\leq 1\big\}.$$

(ii) *There exists a $\mu_K\in\mathcal{M}_+(K)$ such that*

$$\mu_K(K)=\int_{\mathbb{R}^n}\big(R_\alpha^*\mu_K(x)\big)^{p'}\,dx=\int_{\mathbb{R}^{1+n}_+} R_\alpha(R_\alpha^*\mu_K)^{p'-1}\,d\mu_K=C_p^{(\alpha)}(K).$$

Proof. (i) For convenience, put

$$\tilde{C}_p^{(\alpha)}(K)=\sup\big\{\|\mu\|_1^p:\ \mu\in\mathcal{M}_+(K)\ \ \&\ \ \|R_\alpha^*\mu\|_{L^{p'}(\mathbb{R}^n)}\leq 1\big\}.$$

Since

$$\|\mu\|_1=\mu(K)\leq\int_{\mathbb{R}^{1+n}_+} R_\alpha f\,d\mu=\int_{\mathbb{R}^n} f(x)\,R_\alpha^*\mu(x)\,dx\leq\|f\|_{L^p(\mathbb{R}^n)}\|R_\alpha^*\mu\|_{L^{p'}(\mathbb{R}^n)}$$

holds for all compact sets $K\subset\mathbb{R}^{1+n}_+$, one has $\tilde{C}_p^{(\alpha)}(K)\leq C_p^{(\alpha)}(K)$ for any compact set $K\subset\mathbb{R}^{1+n}_+$. Moreover, this last inequality is actually an equality - in fact, if

$$\begin{cases}\mathcal{X}=\{\mu:\ \ \mu\in\mathcal{M}_+(K)\ \&\ \mu(K)=1\};\\ \mathcal{Y}=\Big\{f:\ \ 0\leq f\in L^p(\mathbb{R}^n)\ \&\ \|f\|_{L^p(\mathbb{R}^n)}\leq 1\Big\};\\ \mathcal{Z}=\Big\{f:\ \ 0\leq f\in L^p(\mathbb{R}^n)\ \&\ R_\alpha f\geq 1_K\Big\};\\ \mathsf{E}(\mu,f)=\int_{\mathbb{R}^n}(R_\alpha^*\mu)(x)f(x)\,dx=\int_{\mathbb{R}^{1+n}_+} R_\alpha f\,d\mu,\end{cases}$$

then an easy computation, along with an application of [3, Theorem 2.4.1], gives

$$\begin{aligned}\min_{\mu\in\mathcal{M}_+(K)}\frac{\|R_\alpha^*\mu\|_{L^{p'}(\mathbb{R}^n)}}{\mu(K)}&=\min_{\mu\in\mathcal{X}}\sup_{f\in\mathcal{Y}}\mathsf{E}(\mu,f)=\sup_{f\in\mathcal{Y}}\min_{\mu\in\mathcal{X}}\mathsf{E}(\mu,f)\\&=\sup_{0\leq f\in L^p(\mathbb{R}^n)}\frac{\min_{(t,x)\in K} R_\alpha f(t,x)}{\|f\|_{L^p(\mathbb{R}^n)}}=\sup_{f\in\mathcal{Z}}\|f\|^{-1}_{L^p(\mathbb{R}^n)}\\&=\big(C_p^{(\alpha)}(K)\big)^{-\frac{1}{p}},\end{aligned}$$

and hence $\tilde{C}_p^{(\alpha)}(K)\geq C_p^{(\alpha)}(K)$, thereby the desired equality of (i) follows.

Next, let us verify (ii). According to (i), we may select a sequence $\{\mu_j\}$ in $\mathcal{M}_+(K)$ such that

$$\begin{cases} \|R_\alpha^*\mu_j\|_{L^{p'}(\mathbb{R}^n)} = 1; \\ \lim_{j\to\infty}\mu_j(K) = C_p^{(\alpha)}(K); \\ \mu_j \text{ has a weak * limit } \mu \in \mathcal{M}_+(K). \end{cases}$$

Then $\mu(K)^p = C_p^{(\alpha)}(K)$. Note that $R_\alpha^*\mu$ is lower semicontinuous on $\mathcal{M}_+(K)$. So $\|R_\alpha^*\mu\|_{L^{p'}(\mathbb{R}^n)} \le 1$, and consequently its equality holds thanks to (i).

Setting $\mu_K = C_p^{(\alpha)}(K)^{\frac{1}{p'}}\mu$ yields

$$\mu_K(K) = \int_{\mathbb{R}^n} \big(R_\alpha^*\mu_K(x)\big)^{p'}\,dx = C_p^{(\alpha)}(K).$$

Assume that f_K is the function in the definition of $C_p^{(\alpha)}(K)$ obeying

$$\begin{cases} \|f_K\|^p_{L^p(\mathbb{R}^n)} = C_p^{(\alpha)}(K) \\ \text{and} \\ R_\alpha f_K \ge 1 \quad \text{on} \quad K. \end{cases}$$

Using the just-proved (i) we get

$$\mu_K\big(\{(t,x)\in K:\ R_\alpha f_K(t,x) < 1\}\big) = 0$$

whence finding

$$R_\alpha(f_K) = R_\alpha(R_\alpha^*\mu_K)^{p'-1} \ge 1 \quad \text{a.e.} \quad \mu_K \quad \text{on} \quad K.$$

So, a combination of the Fubini theorem and the Hölder inequality derives

$$\begin{aligned} C_p^{(\alpha)}(K) &= \mu_K(K) \le \int_{\mathbb{R}^{1+n}_+} R_\alpha f_K\,d\mu_K \\ &= \int_{\mathbb{R}^n} (R_\alpha^*\mu_K) f_K\,dx \le \|R_\alpha^*\mu_K\|_{L^{p'}(\mathbb{R}^n)}\|f_K\|_{L^p(\mathbb{R}^n)} \\ &= C_p^{(\alpha)}(K)^{\frac{1}{p'}} C_p^{(\alpha)}(K)^{\frac{1}{p}} = C_p^{(\alpha)}(K). \end{aligned}$$

This in turn implies $f_K = (R_\alpha^*\mu_K)^{\frac{1}{p-1}}$, whence completing the argument for (ii). □

2.2. **Basic capacitary properties.** The first basic property of the above-defined L^p type capacity is stated as below.

Proposition 2.
(i) $C_p^{(\alpha)}(\emptyset) = 0$.
(ii) $K_1 \subseteq K_2 \subset \mathbb{R}^{1+n}_+ \Rightarrow C_p^{(\alpha)}(K_1) \le C_p^{(\alpha)}(K_2)$.
(iii)

$$C\Big(\bigcup_{j=1}^\infty K_j\Big) \le \sum_{j=1}^\infty C_p^{(\alpha)}(K_j)$$

for any sequence $\{K_j\}_{j=1}^\infty$ *of subsets of* $\mathbb{R}^{n+1}_+$.
(iv) $C_p^{(\alpha)}\big(K + (0,x_0)\big) = C_p^{(\alpha)}(K)$ *for any* $K \subset \mathbb{R}^{n+1}_+$ *and any* $x_0 \in \mathbb{R}^n$.

Proof. (i) This is trivial.

(ii) This follows from Definiton 2.1.

(iii) If f_j is chosen with $R_\alpha f_j \ge 1$ on K_j, then $f = \sup_{j=1,2,3,\dots} f_j$ satisfies $R_\alpha f \ge 1$ on $\bigcup_{j=1}^\infty K_j$ and

$$\|f\|^p_{L^p(\mathbb{R}^n)} \le \sum_{j=1}^\infty \|f_j\|^p_{L^p(\mathbb{R}^n)}$$

So, the desired inequality follows.

(iv) This is a consequence of the following implication:

$$f_{x_0}(x) = f(x+x_0) \Longrightarrow \|f_{x_0}\|_{L^p(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)}.$$

□

The second basic property of $C_p^{(\alpha)}$ is an equivalent estimate for the capacity of an α-parabolic ball in $\mathbb{R}^{1+n}_+$.

Proposition 3. *For $t_0\ge 0$, $x_0\in\mathbb{R}^n$ and $r_0>0$, call*

$$B^{(\alpha)}_{r_0}(t_0,x_0)=\{(t,x)\in\mathbb{R}^{1+n}_+:\quad r_0^{2\alpha}<t-t_0<2r_0^{2\alpha}\ \text{ and }\ |x-x_0|<r_0/2\}$$

the α-parabolic ball centered at (t_0,x_0) with radius r_0. If $1\le p<\infty$, then

$$C_p^{(\alpha)}\big(B^{(\alpha)}_{r_0}(0,0)\big)=C_p^{(\alpha)}\big(B^{(\alpha)}_{1}(0,0)\big)r_0^n\quad\forall\quad r_0>0.$$

Generally,

$$C_p^{(\alpha)}\big(B^{(\alpha)}_{r_0}(t_0,x_0)\big)\approx r_0^n\quad\forall\quad r_0>0.$$

Proof. If $f\ge 0$ obeys $R_\alpha f\ge 1_{B^{(\alpha)}_{r_0}(0,0)}$, then

$$t=r_0^{2\alpha}s\ \ \&\ \ x=r_0y\ \ \&\ \ f_{r_0}(x)=f(r_0x)$$

ensures

$$(t,x)\in B^{(\alpha)}_{r_0}(0,0)\Longleftrightarrow(s,y)\in B^{(\alpha)}_{1}(0,0)$$

and

$$R_\alpha f(t,x)\ge 1\ \ \forall\ \ (t,x)\in B^{(\alpha)}_{r_0}(0,0)\Longleftrightarrow R_\alpha f_{r_0}(s,y)\ge 1\ \ \forall\ \ (s,y)\in B^{(\alpha)}_{1}(0,0).$$

So

$$C_p^{(\alpha)}(B^{(\alpha)}_{r_0}(0,0))=r_0^nC_p^{(\alpha)}(B^{(\alpha)}_{1}(0,0)).$$

Next, let us handle the general case. For $p\in[1,\infty)$, choose $\tilde p,\tilde q$ so that

$$\begin{cases}1\le p\le\tilde p<\frac{np}{n-\min\{n,2\alpha\}};\\ \frac{1}{\tilde q}=\big(\frac{n}{2\alpha}\big)\big(\frac1p-\frac1{\tilde p}\big).\end{cases}$$

If

$$0\le f\in L^p(\mathbb{R}^n)\ \ \&\ \ R_\alpha f(t,x)\ge 1\ \ \forall\ \ (t,x)\in B^{(\alpha)}_{r_0}(t_0,x_0)\subset\mathbb{R}^{1+n}_+,$$

then an application of Lemma 1.1 (i) gives

$$r_0^{\frac{2\alpha}{\tilde q}+\frac{n}{\tilde p}}\lesssim\left(\int_{r_0^{2\alpha}<t-t_0<2r_0^{2\alpha}}\Big(\int_{|x-x_0|<r_0/2}|R_\alpha f(t,x)|^{\tilde p}\,dx\Big)^{\frac{\tilde q}{\tilde p}}dt\right)^{\frac{1}{\tilde q}}\lesssim\|f\|_{L^p(\mathbb{R}^n)},$$

and hence

$$r_0^n\lesssim C_p^{(\alpha)}\big(B^{(\alpha)}_{r_0}(t_0,x_0)\big).$$

For the reverse estimate, choose $f=1_{\{x\in\mathbb{R}^n:\ |x-x_0|<r_0/2\}}$. Note that [24, Proposition 1 (2)] gives two constants κ_1,κ_2 (depending only on α and n) such that

$$\inf_{|x|\le\kappa_1t^{\frac{1}{2\alpha}}}K_t^{(\alpha)}(x)\ge\kappa_2t^{-\frac{n}{2\alpha}}\quad\forall\quad t>0.$$

So, when $|x-x_0|<r_0/2$ and $|y-x_0|<r_0/2$, one has $|x-y|<r_0$ and consequently,

$$R_\alpha f(t,x)=\int_{\{y\in\mathbb{R}^n:\ |y-x_0|<r_0/2\}}K_t^{(\alpha)}(x-y)\,dy\gtrsim\kappa_2\Big(\frac{r_0}{\kappa_1}\Big)^{-n}r_0^n\gtrsim 1$$

holds for any $(t,x)\in B^{(\alpha)}_{r_0}(t_0,x_0)$. As a result, we get

$$C_p^{(\alpha)}\big(B^{(\alpha)}_{r_0}(t_0,x_0)\big)\lesssim r_0^n\quad\forall\quad r_0>0.$$

So, the required equivalence estimate follows. □

2.3. **Linking to Hausdorff capacity.** To handle this issue, let us define the so-called $L^{(\alpha)}$-based Hausdorff capacity, Hausdorff measure, and Hausdorff dimension. For

$$\begin{cases} 0<\varepsilon\le\infty; \\ 0<d<\infty; \\ K\subset\mathbb{R}^{1+n}_+; \\ B^{(\alpha)}_r(t,x)=\big\{(s,y)\in\mathbb{R}^{1+n}_+:\ r^{2\alpha}<s-t<2r^{2\alpha}\ \&\ |y-x|<r/2\big\}; \\ (t,x,r)\in[0,\infty)\times\mathbb{R}^n\times\mathbb{R}_+, \end{cases}$$

let

$$H^{d,\alpha}_\varepsilon(K)=\inf\left\{\sum_{j=1}^\infty r_j^d:\ K\subseteq\bigcup_{j=1}^\infty B^{(\alpha)}_{r_j}(t_j,x_j);\quad \text{with}\quad r_j<\varepsilon\right\}$$

be the d-dimensional $L^{(\alpha)}$-based ε-Hausdorff capacity of K. Then the d-dimensional $L^{(\alpha)}$-based Hausdorff measure of K is defined by

$$H^{d,\alpha}(K)=\lim_{\varepsilon\to0}H^{d,\alpha}_\varepsilon(K).$$

Consequently,

$$\dim^{(\alpha)}_H(K)=\inf\{d:H^{d,\alpha}(K)=0\},$$

the $L^{(\alpha)}$-based Hausdorff dimension of K.

As a kind of the isocapacitary inequality, below is an immediate consequence of Propositons 2 & 3 and Lemma 1.1 (i).

Corollary 1.
(i) *Let K be a bounded Borel subset of $\mathbb{R}^{1+n}_+$. If $1\le p<\infty$, then*

$$C^{(\alpha)}_p(K)\lesssim H^{n,\alpha}_\infty(K).$$

(ii) *Let $A\subset\mathbb{R}_+$ and $B\subset\mathbb{R}^n$ be bounded Borel sets. If*

$$\begin{cases} 1\le p\le\tilde{p}<\frac{np}{n-\min\{n,2\alpha\}}; \\ \frac{1}{\tilde{q}}=\big(\frac{n}{2\alpha}\big)\big(\frac1p-\frac1{\tilde{p}}\big), \end{cases}$$

then

$$\mathscr{L}^1(A)^{\frac{p}{\tilde{q}}}\mathscr{L}^n(B)^{\frac{p}{\tilde{p}}}\lesssim C^{(\alpha)}_p(A\times B)\lesssim H^{n,\alpha}_\infty(A\times B)$$

for some $\epsilon>0$. Here $\mathscr{L}^k(E)$ stands for the k-dimensional Lebesgue measure of E.

3. $L^q(\mathbb{R}^{1+n}_+)$-extensions of $L^p(\mathbb{R}^n)$ via $L^{(\alpha)}$.

3.1. **Capacitary weak and strong type inequalities.** In accordance with the definition of $C^{(\alpha)}_p$, if $L^p_+(\mathbb{R}^n)$ stands for the class of all nonnegative functions in $L^p(\mathbb{R}^n)$ then we always have the following capacitary weak type inequality

$$\lambda^pC^{(\alpha)}_p\big(\{(t,x)\in\mathbb{R}^{1+n}_+:\ R_\alpha f(t,x)\ge\lambda\}\big)\le\|f\|^p_{L^p(\mathbb{R}^n)}\quad\forall\quad f\in L^p_+(\mathbb{R}^n).$$

In fact, we can establish the following capacitary strong type inequality.

Lemma 3.1. *Let $p\in(1,\infty)$. Then*

$$\int_0^\infty C^{(\alpha)}_p\big(\{(t,x)\in\mathbb{R}^{1+n}_+:\ R_\alpha f(t,x)\ge\lambda\}\big)\,d\lambda^p\lesssim\|f\|^p_{L^p(\mathbb{R}^n)}\quad\forall\quad f\in L^p_+(\mathbb{R}^n).$$

Here and henceforth $d\lambda^p=p\lambda^{p-1}d\lambda$.

Proof. Since the desired strong type estimate follows from the density of $C^\infty_0(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$, we are about to verify the result for any nonnegative $C^\infty_0(\mathbb{R}^n)$-function. Following the argument for [3, Theorem 7.1.1], for each $j=0,\pm1,\pm2,...$ and $0\le f\in C^\infty_0(\mathbb{R}^n)$ we set

$$E_j=\{(t,x)\in\mathbb{R}^{1+n}_+:\ R_\alpha f(t,x)\ge2^j\}.$$

If μ_j stands for the measure obtained in Proposition 1 (ii) for E_j, then

$$\begin{aligned} S &= \sum_{j=-\infty}^{\infty} 2^{jp}\mu_j(\mathbb{R}^{1+n}_+) \le \sum_{j=-\infty}^{\infty} 2^{j(p-1)}\int_{\mathbb{R}^{1+n}_+} R_\alpha f\, d\mu_j \\ &= \sum_{j=-\infty}^{\infty} 2^{j(p-1)}\int_{\mathbb{R}^n} f(x) R^*_\alpha\mu_j(x)\,dx \le \|f\|_{L^p(\mathbb{R}^n)}\Big\|\sum_{j=-\infty}^{\infty} 2^{j(p-1)}R^*_\alpha\mu_j\Big\|_{L^{p'}(\mathbb{R}^n)} \\ &= \|f\|_{L^p(\mathbb{R}^n)} T^{\frac{1}{p'}}. \end{aligned}$$

Upon letting

$$\begin{cases} k = 0, \pm1, \pm2, ...; \\ \sigma_k(x) = \sum_{j=-\infty}^{k} 2^{j(p-1)}R^*_\alpha\mu_j(x); \\ \sigma(x) = \sum_{j=-\infty}^{\infty} 2^{j(p-1)}R^*_\alpha\mu_j(x), \end{cases}$$

we have

$$\sigma_k \in L^{p'}(\mathbb{R}^n) \quad \& \quad \lim_{k\to-\infty}\sigma_k = \sigma.$$

In what follows, we are about to prove $T \lesssim S$ through two cases.

Case 1: $2 \le p < \infty$. Note that

$$\sigma(x)^{p'} = p'\sum_{k=-\infty}^{\infty}\sigma_k(x)^{p'-1}2^{k(p-1)}R^*_\alpha\mu_k(x) \quad \text{for a.e.} \quad x \in \mathbb{R}^n.$$

So, an application of the Hölder inequality gives

$$T \le p' T_1^{2-p'} T_2^{p'-1} \lesssim S,$$

where

$$T_1 = \int_{\mathbb{R}^n}\sum_k 2^{kp}\big(R^*_\alpha\mu_k(x)\big)^{p'}\,dx = \sum_k 2^{kp}C^{(\alpha)}_p(E_k) = S$$

and

$$\begin{aligned} T_2 &= \int_{\mathbb{R}^n}\sum_k \sigma_k(x)2^k\big(R^*_\alpha\mu_k(x)\big)^{p'-1}\,dx \\ &= \sum_k\sum_{j\ge k} 2^{j(p-1)+k}\int_{\mathbb{R}^n}\big(R^*_\alpha\mu_j(x)\big)\big(R^*_\alpha\mu_k(x)\big)^{p'-1}\,dx \\ &\lesssim \sum_k\sum_{j\ge k} 2^{j(p-1)+k}C^{(\alpha)}_p(E_j) \\ &\approx \sum_k 2^{kp}C^{(\alpha)}_p(E_k) \approx S. \end{aligned}$$

Case 2: $2 > p > 1$. Similarly we obtain

$$\sigma(x)^{p'} = p'\sum_{k=-\infty}^{\infty}\sigma_k(x)^{p'-1}2^{k(p-1)}R^*_\alpha\mu_k(x) \quad \text{for a.e.} \quad x \in \mathbb{R}^n$$

whence getting

$$\begin{aligned} T &= p'\sum_{k=-\infty}^{\infty}2^{k(p-1)}\int_{\mathbb{R}^n}\sigma_k(x)^{p'-1}R^*_\alpha\mu_k(x)\,dx \\ &\lesssim \sum_{k=-\infty}^{\infty}2^{k(p-1)}\left(\sum_{j=-\infty}^{k}2^{j(p-1)}\Big(\int_{\mathbb{R}^n}\big(R^*_\alpha\mu_j(x)\big)^{p'-1}\big(R^*_\alpha\mu_k(x)\big)\,dx\Big)^{\frac{1}{p'-1}}\right)^{p'-1} \\ &\lesssim \sum_{k=-\infty}^{\infty}2^{kp}C^{(\alpha)}_p(E_k) \approx S. \end{aligned}$$

In all, we have

$$S = \sum_{j=-\infty}^{\infty}2^{jp}C^{(\alpha)}_p(E_j) \lesssim \|f\|^p_{L^p(\mathbb{R}^n)}$$

thereby reaching the desired strong type inequality for $0 \le f \in C^\infty_0(\mathbb{R}^n)$. □

3.2. **The lower sector case $1 < p \le q < \infty$.** In what follows, $\mu \in \mathcal{M}_+(\mathbb{R}^{1+n}_+)$ represents the class of all nonnegative Randon measures on $\mathbb{R}^{1+n}_+$. For $\lambda > 0$ define

$$\kappa(\mu;\lambda) = \inf\{C^{(\alpha)}_p(K) : \text{ compact } K \subset \mathbb{R}^{1+n}_+ \ \&\ \mu(K) \ge \lambda\}.$$

Theorem 3.2. *Let $1 < p \le q < \infty$ and $\mu \in \mathcal{M}_+(\mathbb{R}^{1+n}_+)$. Then the extension $R_\alpha : L^p(\mathbb{R}^n) \mapsto L^q(\mathbb{R}^{1+n}_+, \mu)$ is bounded if and only if*

$$\sup_{\lambda \in \mathbb{R}_+} \lambda^{\frac{p}{q}} / \kappa(\mu;\lambda) < \infty.$$

In particular, if $1 < p < q < \infty$, then $\lambda^{\frac{p}{q}} \lesssim \kappa(\mu;\lambda)\ \forall\ \lambda \in \mathbb{R}_+$ can be replaced by $\mu(B^{(\alpha)}_r(t_0,x_0)) \lesssim r^{qn/p}\ \ \forall\ (r,t_0,x_0) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n$.

Proof. Suppose $R_\alpha : L^p(\mathbb{R}^n) \mapsto L^q(\mathbb{R}^{1+n}_+, \mu)$ is bounded. Then, for a given compact $K \subset \mathbb{R}^{1+n}_+$ we use Proposition 1 (ii) and Hölder's inequality with

$$(p', q') = \Big(\frac{p}{p-1}, \frac{q}{q-1}\Big)$$

to derive

$$\int_{\mathbb{R}^n} f R^*_\alpha \mu_K = \int_{\mathbb{R}^{1+n}_+} R_\alpha f \, d\mu_K \le \|R_\alpha f\|_{L^q(\mathbb{R}^{1+n}_+,\mu)} \mu(K)^{\frac{1}{q'}} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \mu(K)^{\frac{1}{q'}},$$

and consequently,

$$\|R^*_\alpha \mu_K\|_{L^{p'}(\mathbb{R}^n)} \lesssim \mu(K)^{\frac{1}{q'}}.$$

This estimate gives that if

$$E_\lambda(f) = \{(t,x) \in \mathbb{R}^{1+n}_+ : \ |R_\alpha f(t,x)| \ge \lambda\}$$

then

$$\lambda \mu(E_\lambda(f)) \le \int_{\mathbb{R}^{1+n}_+} |R_\alpha| \, d\mu_{E_\lambda} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \|R^*_\alpha \mu_{E_\lambda}\|_{L^{p'}(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \mu(E_\lambda)^{\frac{1}{p'}}$$

and hence

$$\sup_{\lambda \in \mathbb{R}_+} \lambda^q \mu(E_\lambda(f)) \lesssim \|f\|^q_{L^p(\mathbb{R}^n)}.$$

This, upon choosing a function $f \in L^p(\mathbb{R}^n)$ such that $R_\alpha f \ge 1$ on a given compact $K \subset \mathbb{R}^{1+n}_+$, derives

$$\mu(K)^{\frac{1}{q}} \lesssim C^{(\alpha)}_p(K)^{\frac{1}{p}},$$

equivalently,

$$\sup_{\lambda \in \mathbb{R}_+} \lambda^{\frac{p}{q}} / \kappa(\mu;\lambda) < \infty.$$

Conversely, assume that the last condition is true, i.e., the last but one is valid for any compact $K \subset \mathbb{R}^{1+n}_+$. Thus, an application of Lemma 3.1 and the capacitary weak type inequality implies that

$$\begin{aligned}
\int_{\mathbb{R}^{1+n}_+} |R_\alpha f|^q \, d\mu &= \int_0^\infty \mu(E_\lambda) \, d\lambda^q \\
&\lesssim \int_0^\infty C^{(\alpha)}_p(E_\lambda)^{\frac{q}{p}} \, d\lambda^q \\
&\lesssim \|f\|^{q-p}_{L^p(\mathbb{R}^n)} \int_0^\infty C^{(\alpha)}_p(E_\lambda) \, d\lambda^p \\
&\lesssim \|f\|^q_{L^p(\mathbb{R}^n)}
\end{aligned}$$

holds for any $f \in C^\infty_0(\mathbb{R}^n)$, and then for $f \in L^p(\mathbb{R}^n)$ via approximating with $C^\infty_0(\mathbb{R}^n)$-functions.

Next, we verify that under $1 < p < q < \infty$ the criterion $\lambda^{\frac{p}{q}} \lesssim \kappa(\mu;\lambda)$ can be replaced by an easily-checked condition $\mu(B^{(\alpha)}_r(t_0,x_0)) \lesssim r^{qn/p}$.

The implication

$$\lambda^{\frac{p}{q}} \lesssim \kappa(\mu;\lambda) \Longrightarrow \mu(B^{(\alpha)}_r(t_0,x_0)) \lesssim r^{qn/p}$$

follows immediately from Proposition 3. For the reverse implication, we employ [24, Proposition 1 (2)] once again to get that if $(t,x) \in B^{(\alpha)}_r(t_0,x_0)$, $(t_0,x_0) \in \mathbb{R}^{1+n}_+$ and μ_K is as in the previous argument, then a further application of [24, Proposition 1 (2)] gives $K_t^{(\alpha)}(x_0-x)\gtrsim r^{-n}$. This, plus Fubini's theorem, implies

$$\begin{aligned}R_\alpha^*\mu_K(x_0)&=\int_{\mathbb{R}^{1+n}_+}K_t^{(\alpha)}(x_0-x)\,d\mu_K\\&\approx\int_{\mathbb{R}^{1+n}_+}\left(\int_{\left(K_t^{(\alpha)}(x_0-x)\right)^{-1/n}}^\infty\frac{dr}{r^{1+n}}\right)d\mu_K\\&\lesssim\int_{\mathbb{R}^{1+n}_+}\left(\int_0^\infty 1_{B_r^{(\alpha)}(t_0,x_0)}\frac{dr}{r^{1+n}}\right)d\mu_K\\&\approx\int_0^\infty\mu_K\left(B_r^{(\alpha)}(t_0,x_0)\right)\frac{dr}{r^{1+n}}.\end{aligned}$$

Using Minkowski's inequality we get

$$\left\|R_\alpha^*\mu_K\right\|_{L^{p'}(\mathbb{R}^n)}\lesssim\int_0^\infty\left\|\mu_K\left(B_r^{(\alpha)}(t_0,\cdot)\right)\right\|_{L^{p'}(\mathbb{R}^n)}\frac{dr}{r^{1+n}}.$$

In general,

$$\|\mu_K\left(B_r^{(\alpha)}(t_0,\cdot)\right)\|^{p'}_{L^{p'}(\mathbb{R}^n)}\lesssim\mu(K)^{p'-1}\int_{\mathbb{R}^n}\mu_K(B_r^{(\alpha)}(t_0,x_0))\,dx_0\lesssim\mu(K)^{p'}r^n.$$

This implies that for a later-decided number $\delta>0$,

$$\int_\delta^\infty\left\|\mu_K\left(B_r^{(\alpha)}(t_0,\cdot)\right)\right\|_{L^{p'}(\mathbb{R}^n)}\frac{dr}{r^{1+n}}\lesssim\mu(K)\delta^{-\frac{n}{p}}.$$

Meanwhile, using $\mu(B_r^{(\alpha)}(t_0,x_0))\lesssim r^{qn/p}$ we get

$$\|\mu_K\left(B_r^{(\alpha)}(t_0,\cdot)\right)\|^{p'}_{L^{p'}(\mathbb{R}^n)}\lesssim r^{nq(p'-1)/p}\int_{\mathbb{R}^n}\mu_K(B_r^{(\alpha)}(t_0,x_0))\,dx_0\lesssim\mu(K)r^{n(1+\frac{q}{p(p-1)})}$$

thereby obtaining

$$\int_0^\delta\left\|\mu_K\left(B_r^{(\alpha)}(t_0,\cdot)\right)\right\|_{L^{p'}(\mathbb{R}^n)}\frac{dr}{r^{1+n}}\lesssim\mu(K)^{\frac{1}{p'}}\delta^{\frac{n(q-p)}{p^2}}.$$

Now, choosing $\delta=\mu(K)^{\frac{p}{nq}}$ and putting the above estimates together, we find $\|R_\alpha^*\mu_K\|_{L^{p'}(\mathbb{R}^n)}\lesssim\mu(K)^{\frac{1}{q'}}$, whence reaching $\lambda^{\frac{p}{q}}\lesssim\kappa(\mu;\lambda)$. □

Remark 1. Of course, it is more interesting that upon letting $d\mu(t,x)=dtdx$ in Theorem 3.2 we have $\mu(B_r^{(\alpha)}(t_0,x_0))\approx r^{2\alpha+n}$ and thus

$$\|R_\alpha f\|_{L_t^{\tilde q}L_x^{\tilde p}(\mathbb{R}^{1+n}_+)}=\left\|R_\alpha f\right\|_{L^{\tilde q}(\mathbb{R}^{1+n}_+)}\lesssim\|f\|_{L^p(\mathbb{R}^n)}\quad\text{for}\quad\tilde q=\tilde p=p(1+\frac{2\alpha}{n}).$$

This is a special case of Lemma 1.1 (i).

3.3. **The upper sector case $1<q<p<\infty$.**

Theorem 3.3. *Let $1<q<p<\infty$ and $\mu\in\mathcal{M}_+(\mathbb{R}^{1+n}_+)$. Then the extension $R_\alpha:L^p(\mathbb{R}^n)\mapsto L^q(\mathbb{R}^{1+n}_+,\mu)$ is bounded if and only if*

$$\int_0^\infty\left(\lambda^{\frac{p}{q}}/\kappa(\mu;\lambda)\right)^{\frac{q}{p-q}}\lambda^{-1}d\lambda<\infty.$$

Proof. In what follows, let $1<q<p<\infty$. Our argument is inspired by a discretization process in [18].

On the one hand, suppose $R_\alpha:L^p(\mathbb{R}^n)\mapsto L^q(\mathbb{R}^{1+n}_+,\mu)$ is bounded. Then

$$\left(\int_{\mathbb{R}^{1+n}_+}|R_\alpha f|^q d\mu\right)^{\frac{1}{q}}\lesssim\|f\|_{L^p(\mathbb{R}^n)}\ \ \forall\ \ f\in L^p(\mathbb{R}^n).$$

Then

$$\sup_{\lambda>0}\lambda\Big(\mu\big(E_\lambda(f)\big)\Big)^{\frac{1}{q}}\lesssim\|f\|_{L^p(\mathbb{R}^n)}\ \ \forall\ \ f\in L^p(\mathbb{R}^n).$$

For each integer j there are a compact set $K_j \subset \mathbb{R}^{1+n}_+$ and a function $f_j \in L^p(\mathbb{R}^n)$ such that

$$\begin{cases} C_p^{(\alpha)}(K_j) \le 2\kappa(\mu; 2^j); \\ \mu(K_j) > 2^j; \\ R_\alpha f_j \ge 1_{K_j}; \\ \|f_j\|^p_{L^p(\mathbb{R}^n)} \le 2C_p^{(\alpha)}(K_j). \end{cases}$$

For the integers i, k with $i < k$ let

$$f_{i,k} = \sup_{i\le j\le k} \left(\frac{2^j}{\kappa(\mu; 2^j)}\right)^{\frac{1}{p-q}} f_j.$$

Then

$$\|f_{i,k}\|^p_{L^p(\mathbb{R}^n)} \lesssim \sum_{j=i}^{k} \left(\frac{2^j}{\kappa(\mu; 2^j)}\right)^{\frac{p}{p-q}} \|f_j\|^p_{L^p(\mathbb{R}^n)} \lesssim \sum_{j=i}^{k} \left(\frac{2^j}{\kappa(\mu; 2^j)}\right)^{\frac{p}{p-q}} \kappa(\mu; 2^j).$$

Note that for $i \le j \le k$,

$$(t, x) \in K_j \Longrightarrow |R_\alpha f_{i,k}(t, x)| \ge \left(\frac{2^j}{\mathrm{c}_p(\mu; 2^j)}\right)^{\frac{1}{p-q}}.$$

This in turn leads to

$$2^j < \mu(K_j) \le \mu\left(E_{\left(\frac{2^j}{\kappa(\mu;2^j)}\right)^{\frac{1}{p-q}}}(f_{i,k})\right).$$

Hence

$$\begin{aligned} \|f_{i,k}\|^q_{L^p(\mathbb{R}^n)} &\gtrsim \int_{\mathbb{R}^{1+n}_+} |R_\alpha f_{i,k}|^q \, d\mu \\ &\approx \int_0^\infty \Big(\inf\{\lambda :\ \mu\big(E_\lambda(f_{i,k})\big) \le s\}\Big)^q ds \\ &\gtrsim \sum_{j=i}^{k} \Big(\inf\{\lambda :\ \mu\big(E_\lambda(f_{i,k})\big) \le 2^j\}\Big)^q 2^j \\ &\gtrsim \sum_{j=i}^{k} \left(\frac{2^j}{\kappa(\mu; 2^j)}\right)^{\frac{q}{p-q}} 2^j \\ &\gtrsim \left(\frac{\sum_{j=i}^{k} \left(\frac{2^j}{\kappa(\mu;2^j)}\right)^{\frac{q}{p-q}} 2^j}{\left(\sum_{j=i}^{k} \left(\frac{2^j}{\kappa(\mu;2^j)}\right)^{\frac{p}{p-q}} \kappa(\mu; 2^j)\right)^{\frac{q}{p}}}\right) \|f_{i,k}\|^q_{L^p(\mathbb{R}^n)} \\ &\approx \left(\sum_{j=i}^{k} \frac{2^{\frac{jp}{p-q}}}{\big(\kappa(\mu; 2^j)\big)^{\frac{q}{p-q}}}\right)^{\frac{p-q}{p}} \|f_{i,k}\|^q_{L^p(\mathbb{R}^n)}. \end{aligned}$$

This implies

$$\int_0^\infty \Big(\lambda^{p/q}/\kappa(\mu; \lambda)\Big)^{\frac{q}{p-q}} \lambda^{-1} \, d\lambda \lesssim \sum_{j=-\infty}^{\infty} \frac{2^{\frac{jp}{p-q}}}{\big(\kappa(\mu; 2^j)\big)^{\frac{q}{p-q}}} \lesssim 1.$$

On the other hand, suppose

$$I_{p,q}(\mu) = \int_0^\infty \left(\frac{\lambda^{\frac{p}{q}}}{\kappa(\mu; \lambda)}\right)^{\frac{q}{p-q}} \frac{d\lambda}{\lambda} < \infty.$$

Now for each integer $j = 0, \pm1, \pm2, ...$ and $f \in C_0^\infty(\mathbb{R}^n)$, let

$$S_{p,q}(\mu; f) = \sum_{j=-\infty}^{\infty} \frac{\big(\mu\big(E_{2^j}(f)\big) - \mu\big(E_{2^{j+1}}(f)\big)\big)^{\frac{p}{p-q}}}{\Big(C_p^{(\alpha)}\big(E_{2^j}(f)\big)^{\frac{q}{p-q}}}.$$

Using integration-by-part, Hölder's inequality and Lemma 3.1 we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^{1+n}_+} |R_\alpha f|^q \, d\mu \\
&= -\int_0^\infty \lambda^q \, d\mu(E_\lambda(f)) \\
&\lesssim \sum_{j=-\infty}^{\infty} \big(\mu(E_{2^j}(f)) - \mu(E_{2^{j+1}}(f)\big) 2^{jq} \\
&\lesssim \big(S_{p,q}(\mu; f)\big)^{\frac{p-q}{p}} \left(\sum_{j=-\infty}^{\infty} 2^{jp} C_p^{(\alpha)}\big(E_{2^j}(f)\big) \right)^{\frac{q}{p}} \\
&\lesssim \big(S_{p,q}(\mu; f)\big)^{\frac{p-q}{p}} \left(\int_0^\infty C_p^{(\alpha)}\big(\{(t,x) \in \mathbb{R}^{1+n}_+ : \ |R_\alpha f(t,x)| > \lambda\}\big) \, d\lambda^p \right)^{\frac{q}{p}} \\
&\lesssim \big(S_{p,q}(\mu; f)\big)^{\frac{p-q}{p}} \|f\|^q_{L^p(\mathbb{R}^n)}.
\end{aligned}
$$

Note also that

$$
\begin{aligned}
&\big(S_{p,q}(\mu; f)\big)^{\frac{p-q}{p}} \\
&= \left(\sum_{j=-\infty}^{\infty} \frac{\big(\mu(E_{2^j}(f)) - \mu(E_{2^{j+1}}(f))\big)^{\frac{p}{p-q}}}{\Big(C_p^{(\alpha)}\big(E_{2^j}(f)\big)\Big)^{\frac{q}{p-q}}} \right)^{\frac{p-q}{p}} \\
&= \left(\sum_{j=-\infty}^{\infty} \frac{\big(\mu(E_{2^j}(f)) - \mu(E_{2^{j+1}}(f))\big)^{\frac{p}{p-q}}}{\Big(\kappa(\mu; \mu\big(E_{2^j}(f)\big))\Big)^{\frac{q}{p-q}}} \right)^{\frac{p-q}{p}} \\
&= \left(\sum_{j=-\infty}^{\infty} \frac{\big(\mu(E_{2^j}(f)\big)^{\frac{p}{p-q}} - \big(\mu(E_{2^{j+1}}(f)\big)^{\frac{p}{p-q}}}{\Big(\kappa(\mu; \mu\big(E_{2^j}(f)\big))\Big)^{\frac{q}{p-q}}} \right)^{\frac{p-q}{p}} \\
&\lesssim \left(\int_0^\infty \frac{ds^{\frac{p}{p-q}}}{\big(\kappa(\mu; s)\big)^{\frac{q}{p-q}}} \right)^{\frac{p-q}{p}} \\
&\approx \big(I_{p,q}(\mu)\big)^{\frac{p-q}{p}}.
\end{aligned}
$$

Therefore

$$
\left(\int_{\mathbb{R}^{1+n}_+} |R_\alpha f|^q \, d\mu \right)^{\frac{1}{q}} \lesssim \big(I_{p,q}(\mu)\big)^{\frac{p-q}{pq}} \|f\|_{L^p(\mathbb{R}^n)}.
$$

□

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